1 ALE — arbitrary Lagrangian–Eulerian method

1.1 Equations

- spatial configuration R_x (Eulerian)
- material configuration R_X (Lagrangian), material velocity:

$$v(X,t) = \frac{\partial x}{\partial t}|_X \equiv \frac{\mathrm{d}x}{\mathrm{d}t} \tag{1}$$

- **referential configuration** R_χ where reference coordinates χ are introduced to identify the grid points:

$$\Phi : R_\chi \mapsto R_x \tag{2}$$

  mesh velocity:

$$\hat{v}(\chi,t) = \frac{\partial x}{\partial t}|_\chi \tag{3}$$

  convective velocity (the relative velocity between the material and the mesh):

$$c := v - \hat{v} \tag{4}$$

- material time derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}|_x + v \cdot \nabla f \tag{5}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}|_\chi + c \cdot \nabla f \tag{6}$$

- differential forms of conservation equations

$$\frac{\partial \rho}{\partial t}|_\chi + c \cdot \nabla \rho = -\rho \nabla \cdot v \tag{7}$$

$$\rho \left(\frac{\partial v}{\partial t}|_\chi + c \cdot \nabla v \right) = \nabla \cdot \boldsymbol{\sigma} + \rho f \tag{8}$$

- Reynolds transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{Vt} f(x,t) \mathrm{d}V = \int_{Vt} \frac{\partial f(x,t)}{\partial t} \mathrm{d}V + \int_{St} f(x,t) v \cdot n \mathrm{d}S \,, \tag{9}$$

  where V_t is material volume, and in ALE:

$$\frac{\partial}{\partial t}|_\chi \int_{Vt} f(x,t) \mathrm{d}V = \int_{Vt} \frac{\partial f(x,t)}{\partial t} \mathrm{d}V + \int_{St} f(x,t) \hat{v} \cdot n \mathrm{d}S \,, \tag{10}$$

  where boundary S_t of volume V_t moves with mesh velocity $\hat{v}$

- integral forms of conservation equations

$$\frac{\partial}{\partial t}|_\chi \int_{Vt} \rho \mathrm{d}V + \int_{St} \rho c \cdot n \mathrm{d}S = 0 \tag{11}$$

$$\frac{\partial}{\partial t}|_\chi \int_{Vt} \rho v \mathrm{d}V + \int_{St} \rho v c \cdot n \mathrm{d}S = \int_{Vt} \Big(\nabla \cdot \boldsymbol{\sigma} + \rho h \Big) \mathrm{d}V \tag{12}$$

1.2 Mesh update methods

- geometric methods
- Laplacian smoothing — solving Laplace equation for node velocity or position; mesh distortion minimization; mesh elasticity
- mesh adaptation — mesh moves towards zones with high solution gradient

1.3 Applications

- free surface
- fluid–structure interaction
- mesh optimization

1.4 Examples in Elmer

- www.csc.fi/english/pages/elmer/examples
- growth of topography above velocity discontinuity:
  - elastic mesh in x,y
  - elastic mesh in y, fixed in x
  - elastic mesh in y, linear in x
- sand–box experiment

1.5 Literature

J. Donea, A. Huerta, J.–Ph. Ponthot and A. Rodriguez–Ferran: Arbitrary Lagrangian-=Eulerian methods. Encyclopedia of Computational Mechanics, 2004