1 Visco-plastic rheology

1.1 Effective viscosity

Rheology specifies the relationship between the deviatoric part of the stress tensor $\boldsymbol{\sigma}$ and the strain rate tensor $\dot{\boldsymbol{\epsilon}}$,

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\nabla\vec{v} + \nabla^{\mathrm{T}}\vec{v}). \tag{1}$$

We define effective viscosity η_{eff},

$$\boldsymbol{\sigma} = 2\eta_{\mathrm{eff}}\dot{\boldsymbol{\epsilon}}, \tag{2}$$

which includes viscous and plastic components,

$$\frac{1}{\eta_{\mathrm{eff}}} = \frac{1}{\eta_{\mathrm{visc}}} + \frac{1}{\eta_{\mathrm{plast}}}. \tag{3}$$

Also the strain rate tensor can be split into viscous and plastic part,

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_{\mathrm{visc}} + \dot{\boldsymbol{\epsilon}}_{\mathrm{plast}}\ , \tag{4}$$

where

$$\dot{\boldsymbol{\epsilon}}_{\mathrm{visc}} = \frac{1}{2\eta}\boldsymbol{\sigma}, \tag{5}$$

$$\dot{\boldsymbol{\epsilon}}_{\mathrm{plast}} = 0 \qquad \text{if } \sigma_{\mathrm{II}} < \sigma_{\mathrm{yield}} \tag{6}$$

$$= \chi\frac{\boldsymbol{\sigma}}{\sigma_{\mathrm{II}}} \quad \text{if } \sigma_{\mathrm{II}} = \sigma_{\mathrm{yield}}, \tag{7}$$

where σ_{II} is the second invariant of deviatoric stress, σ_{yield} is the plastic yield strength and χ is plastic multiplier satisfying the yield condition $\sigma_{\mathrm{II}} = \sigma_{\mathrm{yield}}$. If we are not interested in the distribution of deformation between viscous and plastic components, we can directly prescribe

$$\eta_{\mathrm{eff}} = \frac{\sigma_{\mathrm{yield}}}{\dot{\epsilon}_{\mathrm{II}}}, \tag{8}$$

so that the yield condition is satisfied.

1.2 Stress state

Stress tensor $\boldsymbol{\tau}$ in basis $\{\vec{e}_x, \vec{e}_z\}$ in 2D has components

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{pmatrix}. \tag{9}$$

Let's express the traction $\vec{T}$ on a plane with a normal vector $\vec{n} = (\cos\beta, \sin\beta)$ and tangent vector $\vec{t}$ in basis $\{\vec{e}_x, \vec{e}_z\}$

$$T_x = \tau_{xx}\cos\beta + \tau_{xz}\sin\beta\,, \tag{10}$$
$$T_z = \tau_{xz}\cos\beta + \tau_{zz}\sin\beta\,, \tag{11}$$

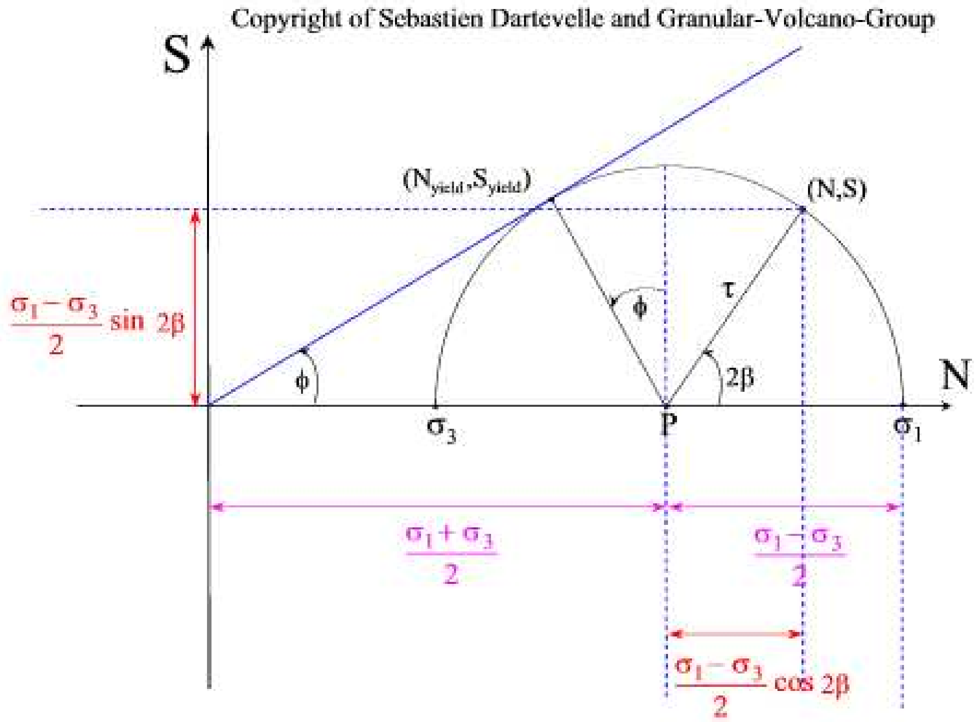
and then in basis $\{\vec{n}, \vec{t}\}$

$$\begin{aligned}
T_{\vec{n}} &= \vec{n}\cdot(T_x, T_z) = && (12)\\
&= T_x\cos\beta + T_z\sin\beta = && (13)\\
&= \tau_{xx}\cos^2\beta + 2\tau_{xz}\cos\beta\sin\beta + \tau_{zz}\sin^2\beta = && (14)\\
&= \frac{\tau_{xx}+\tau_{zz}}{2} + \frac{\tau_{xx}-\tau_{zz}}{2}\cos 2\beta + \tau_{xz}\sin 2\beta && (15)\\
T_{\vec{t}} &= \vec{t}\cdot(T_x, T_z) = && (16)\\
&= -\frac{\tau_{xx}-\tau_{zz}}{2}\sin 2\beta + \tau_{xz}\cos 2\beta\,. && (17)
\end{aligned}$$

In the case $\tau_{xz} = 0$, the principal axes of the stress tensor are parallel to the $\vec{e}_x$, $\vec{e}_z$ coordinates and β is the angle between the principal axis and $\vec{n}$. The values of the principal stresses σ_1 and σ_3 are equal to the eigenvalues of the stress tensor,

$$\sigma_{1,3} = \frac{\tau_{xx}+\tau_{zz}}{2} \pm \sqrt{\left(\frac{\tau_{xx}-\tau_{zz}}{2}\right)^2 + \tau_{xz}^2}\,. \tag{18}$$

The components of traction can be visualized in the Mohr's diagram ($T_{\vec{n}} \equiv N \equiv \sigma$, $T_{\vec{t}} \equiv S \equiv \tau$):



http://www.granular-volcano-group.org/frictional_theory.html

1.3 Yield criterion

From several empirical yield criteria we choose the simplest pressure dependent yield criterion — the Mohr-Coulomb yield criterion. This criterion assumes a linear dependence between shear and normal stresses at yield,

$$T_{\vec{t}} = T_{\vec{n}} \tan\phi + C, \tag{19}$$

where parameter C is called cohesion and parameter ϕ is called effective internal angle of friction. The brittle failure on a plane occurs when $2\beta = \pi/2 + \phi$. Then,

$$T_{\vec{n}} = \frac{\tau_{xx} + \tau_{zz}}{2} + \frac{\tau_{xx} - \tau_{zz}}{2} \sin\beta \tag{20}$$

$$T_{\vec{t}} = \frac{\tau_{xx} - \tau_{zz}}{2} \cos\beta \tag{21}$$

and after introducing mean stress (or pressure) p and differential stress $\Delta\sigma$

$$p = \frac{\sigma_1 + \sigma_3}{2}, \tag{22}$$

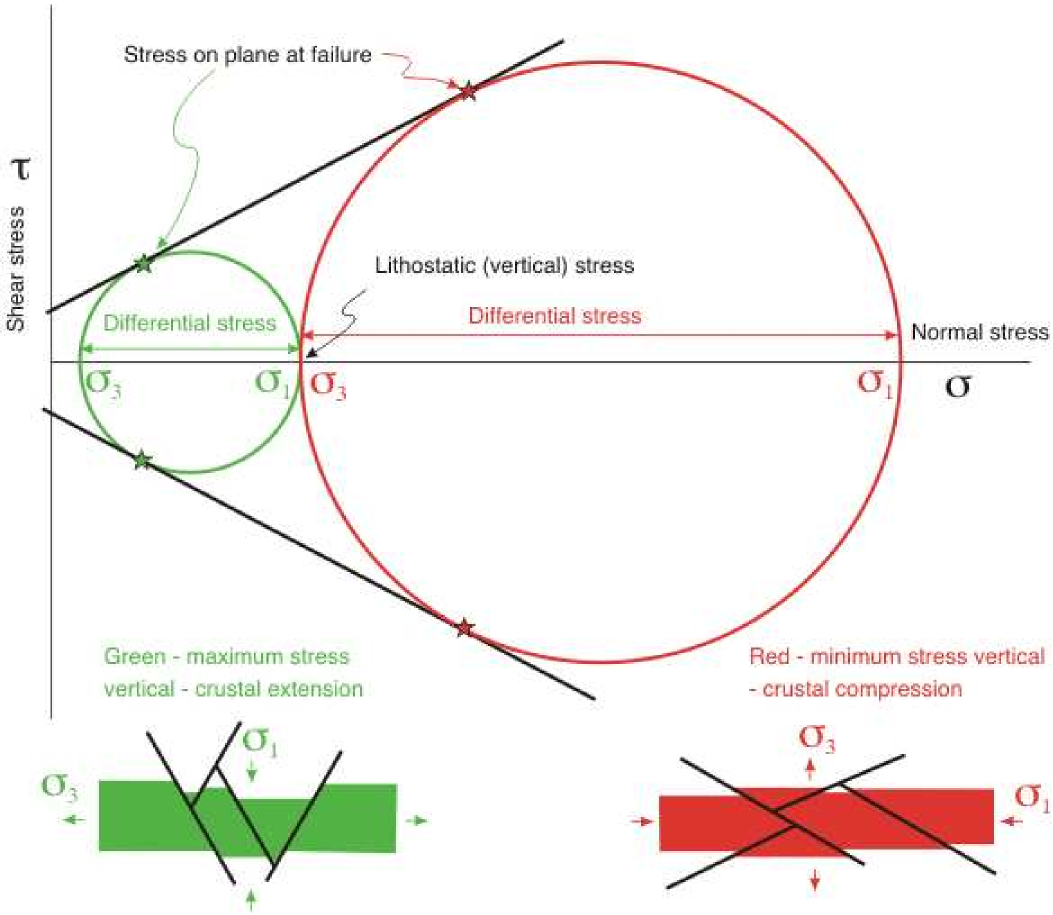
$$\Delta\sigma = \frac{\sigma_1 - \sigma_3}{2}, \tag{23}$$

we can express the Mohr-Coulomb criterion in the form

$$\Delta\sigma = p \sin\phi + C \cos\phi\,. \tag{24}$$

Mohr-Coulomb criterion is in 2D equivalent to the Drucker-Prager yield criterion

$$\sigma_{\mathrm{II}} = p \sin\phi + C \cos\phi\,. \tag{25}$$



http://courses.eas.ualberta.ca/eas421/lecturepages/stressdiags.html#mohnstress