

Particle dynamics in convecting fluids



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Nicola Tosi and Enrico Calzavarini



Content

- Heat budget of the Earth
- Settling of inertial particles in turbulent Rayleigh-Bénard convection
- What next? Discussion...



Radiogenic heat sources (?)

- How much **U, Th, K** in the Earth ?
- ... What is the Earth made of ?

Depends on who you ask

Earth's compositional models

- **O, Fe, Si, Mg** account for **~90%** of Earth's mass
- **+ Al, Ca, Ni, S ... >98%** of Earth's mass
- Negligible K, Th, U in the metallic core
- 10–20–30 ppb of U in **Silicate Earth**
- Th/U ratio of ~4
- K/U ratio of 7000–15000
- **10–20–30 TW** of radiogenic heating

Bulk Silicate Earth (BSE) models:

Low-H
~10 TW

Estatite chondrite models
Collisional erosion models

Middle-H
~20 TW

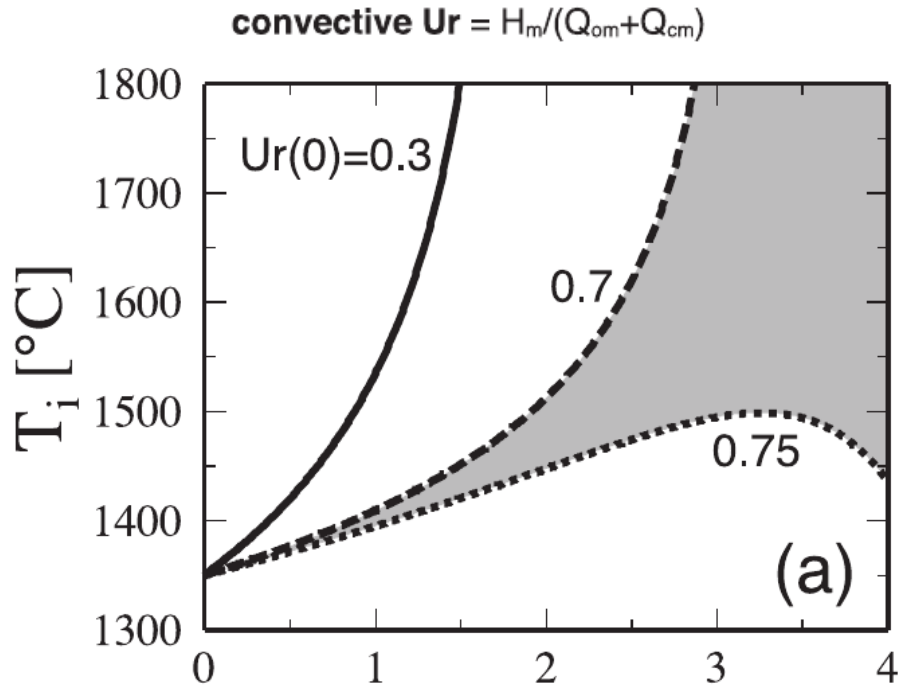
Chondritic + petrological
constraints

High-H
~30 TW

Classical parameterized
convection models

A factor of 3 uncertainty
in the amount of U, Th, K
in the Earth...

Thermal catastrophe



$$C_m \frac{dT_m}{dt} = H + Q_{CMB} - Q_S,$$

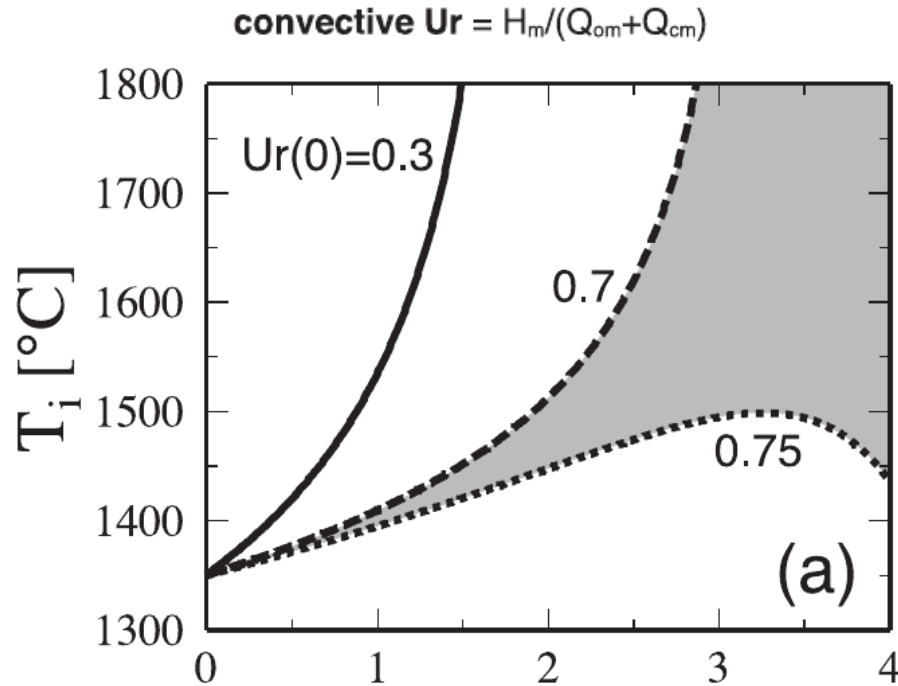
$$Q_S = AS_m \frac{k(T_m - T_s)}{D} Ra^\beta \left(\frac{T_m}{T_m - T_s} \right)^{1+\beta},$$

$$Ra = \frac{g\alpha(T_m - T_s)D^3}{\kappa\nu},$$

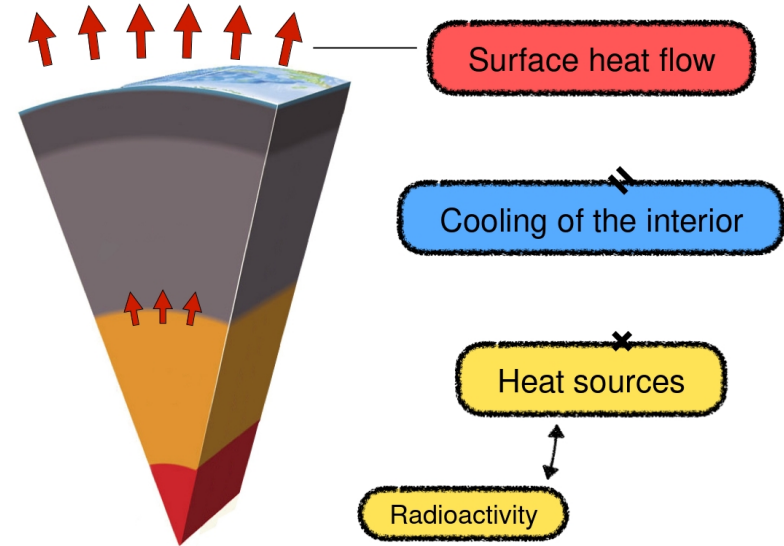
$$Q_S = Q_S^0 \left(\frac{\nu(T_m)}{\nu(T_m^0)} \right)^{-\beta} \left(\frac{T_m}{T_m^0} \right)^{1+\beta}.$$

Korenaga 2008: Urey ratio and Earth's mantle

Thermal catastrophe

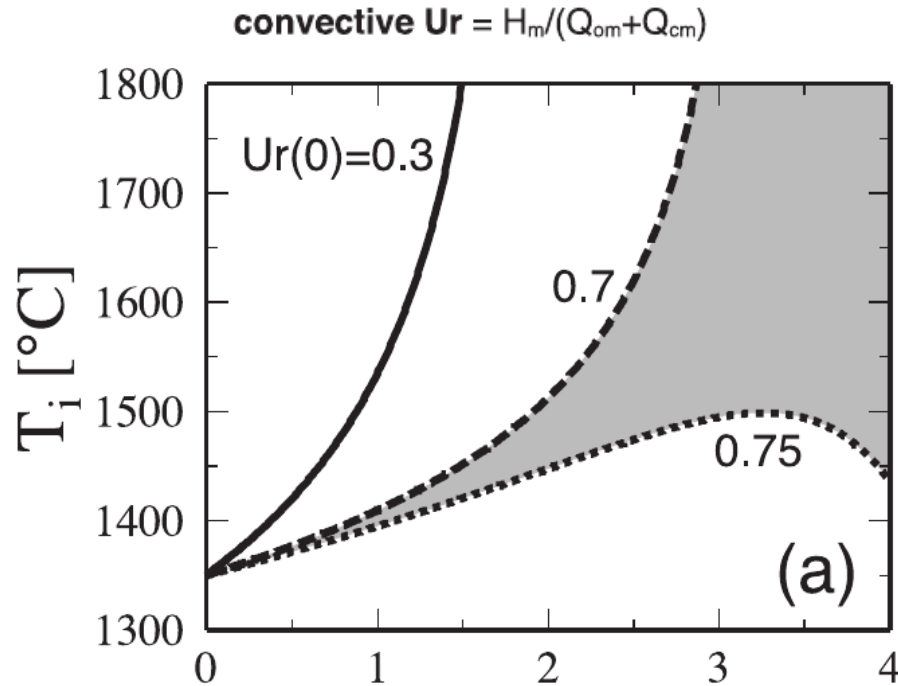


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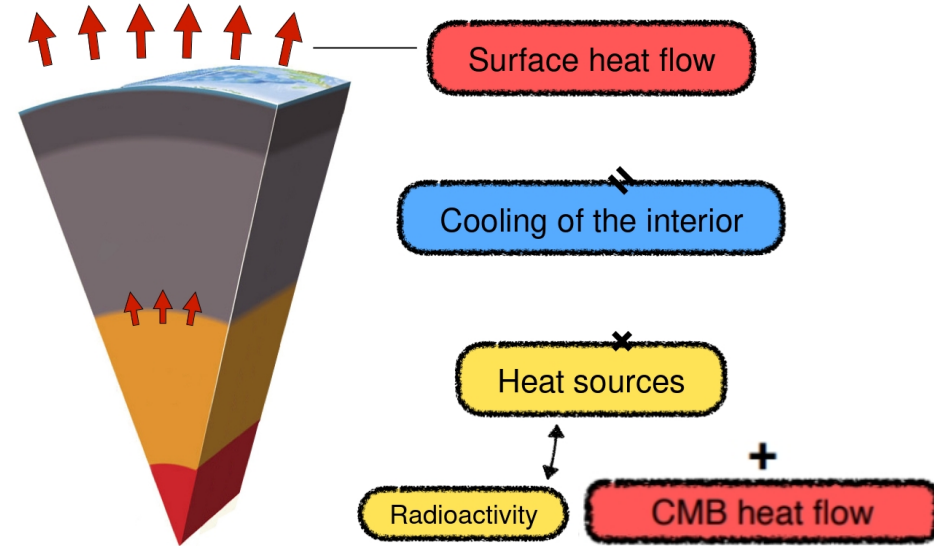


Classical scaling: $Nu \sim Ra^{0.33}$

Thermal catastrophe



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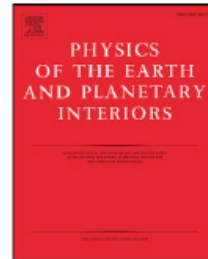
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journal homepage: www.elsevier.com/locate/pepi



Minimum heat flow from the core and thermal evolution of the Earth

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^b Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

^c Department of Astronomy and Astrophysics, Technische Universität, Berlin, Germany

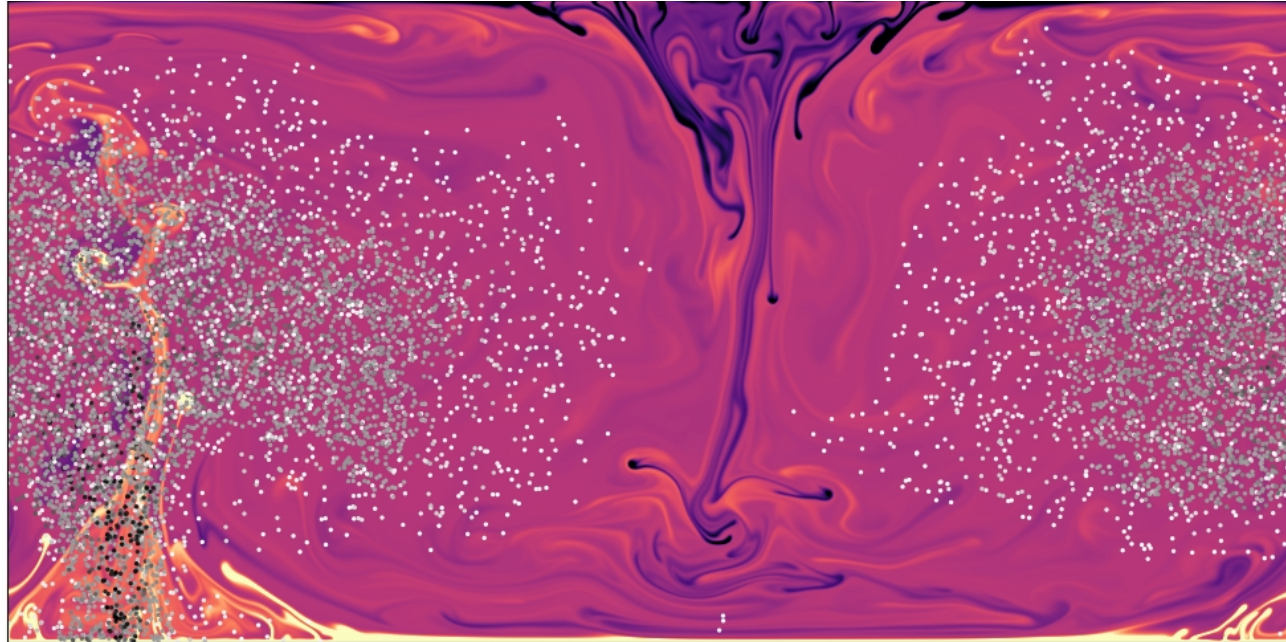


- Dealing with the thermal catastrophe conundrum by including of Q_{CMB}
- Q_{CMB} is the minimum needed to operate Geodynamo (Labrosse, 2015)
- Code (can be made) available

$$C_m \frac{dT_m}{dt} = H + Q_{\text{CMB}} - Q_s,$$

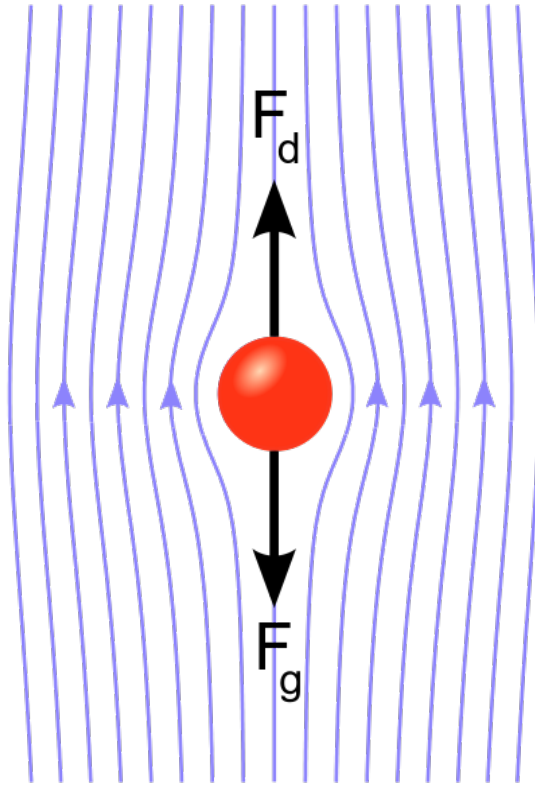
Settling of inertial particles in turbulent Rayleigh-Bénard convection

Vojtěch Patočka ^{1,*} Enrico Calzavarini ² and Nicola Tosi ¹





Flow past particles (focus of this study)

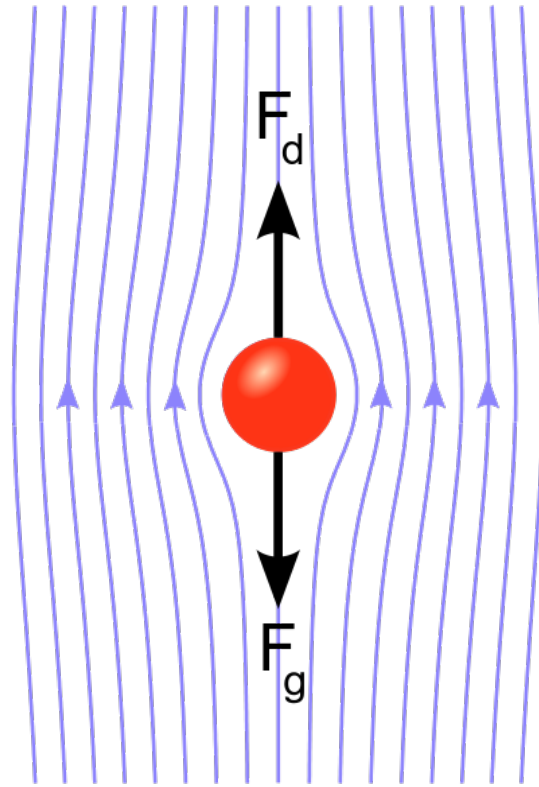


Stokes law:

$$F_d = 6\pi \mu R v$$

$$F_g = (\rho_p - \rho_f) g \frac{4}{3} \pi R^3$$

Flow past particles (focus of this study)



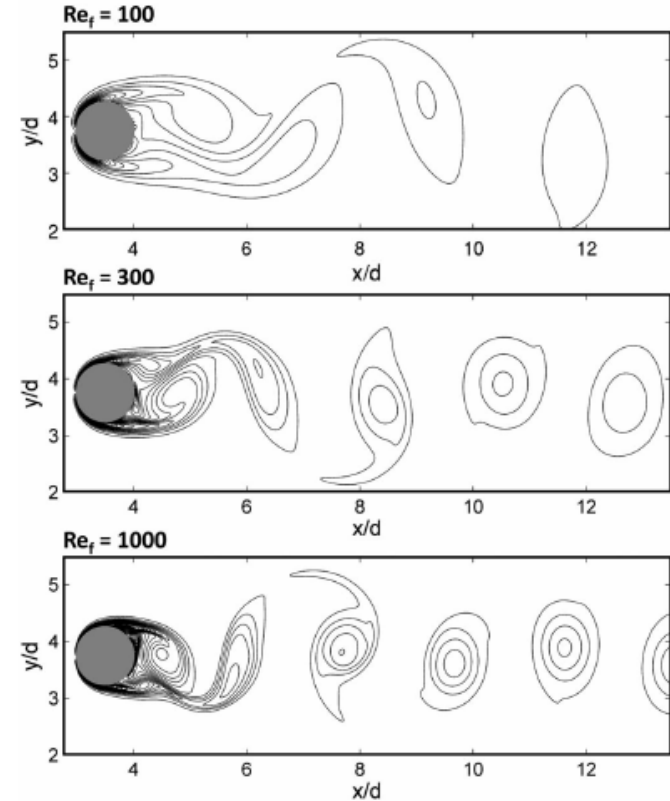
$$Re_p := v_t r_p / \nu$$

← **laminar**, turbulent →

Stokes law:

$$F_d = 6\pi \mu R v$$

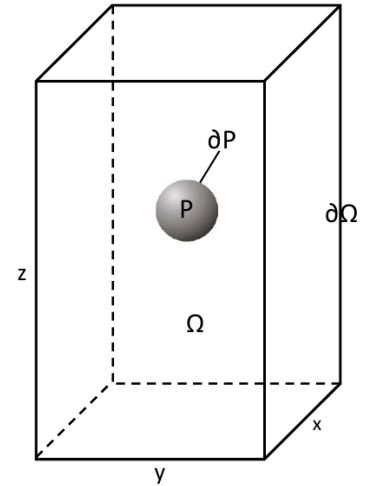
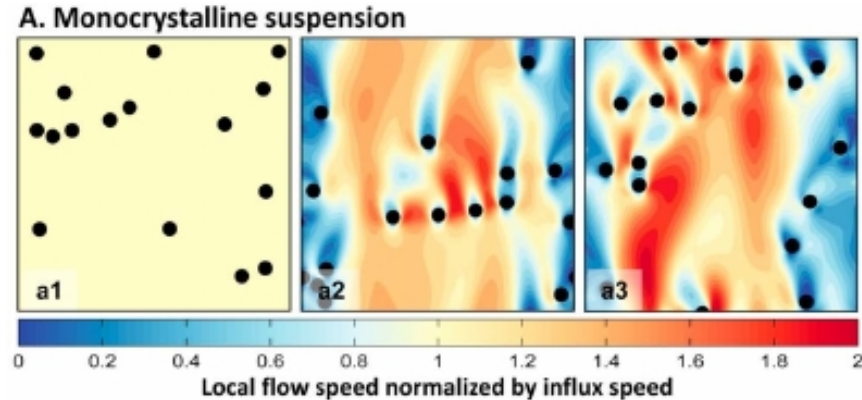
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Suckale et al., 2012

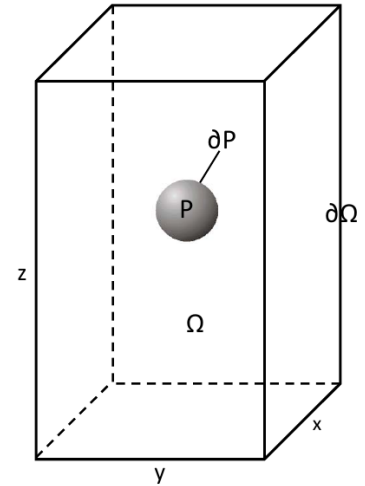
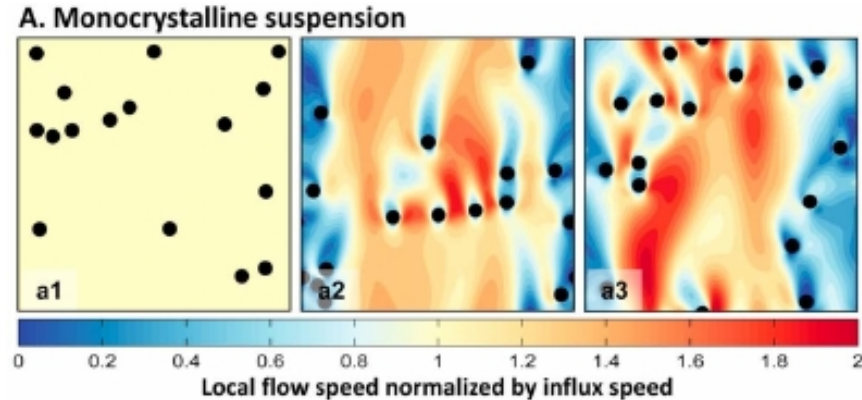
Existing numerical methods

- Macroscopic methods that incorporate particles via Lagrangian tracers
- Semi-analytical and numerical methods based on Basset-Boussinesq-Oseen equation
- Direct numerical simulations of fluid-solid systems
(Verhoeven & Hansen, 2009, Suckale et al., 2012, Maas & Hansen, 2015, 2019)



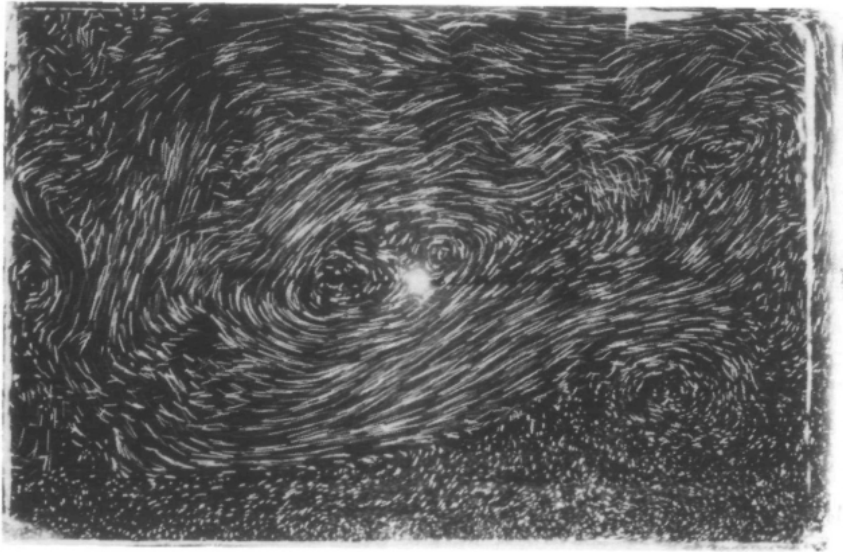
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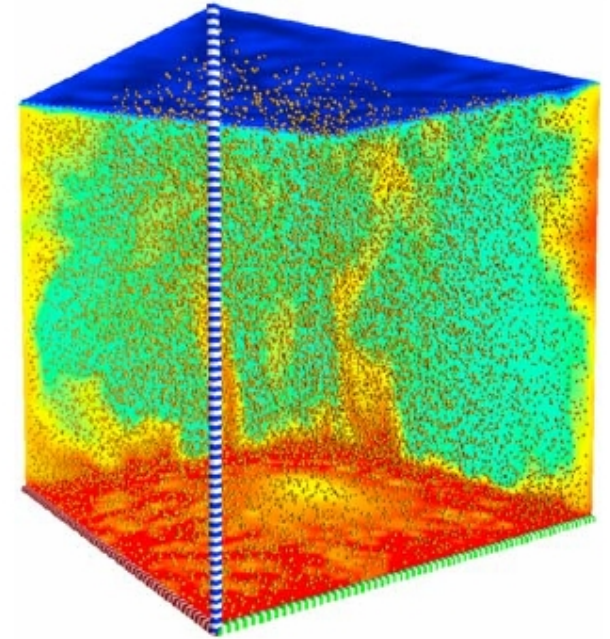
Small particle dilemma

Martin & Nokes, 1989



$$N(t) = N_0 \exp(-v_t t/H)$$

Verhoeven et al., 2009
Maas & Hansen, 2015, 2019



$$N(t) = N_0$$

Numerical model

NVS fluid:

$$\begin{aligned}\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} + \theta \hat{\mathbf{z}} \\ \nabla \cdot \mathbf{u} &= 0 \\ \partial_\tau \theta + (\mathbf{u} \cdot \nabla) \theta &= \frac{1}{\sqrt{PrRa}} \nabla^2 \theta,\end{aligned}$$

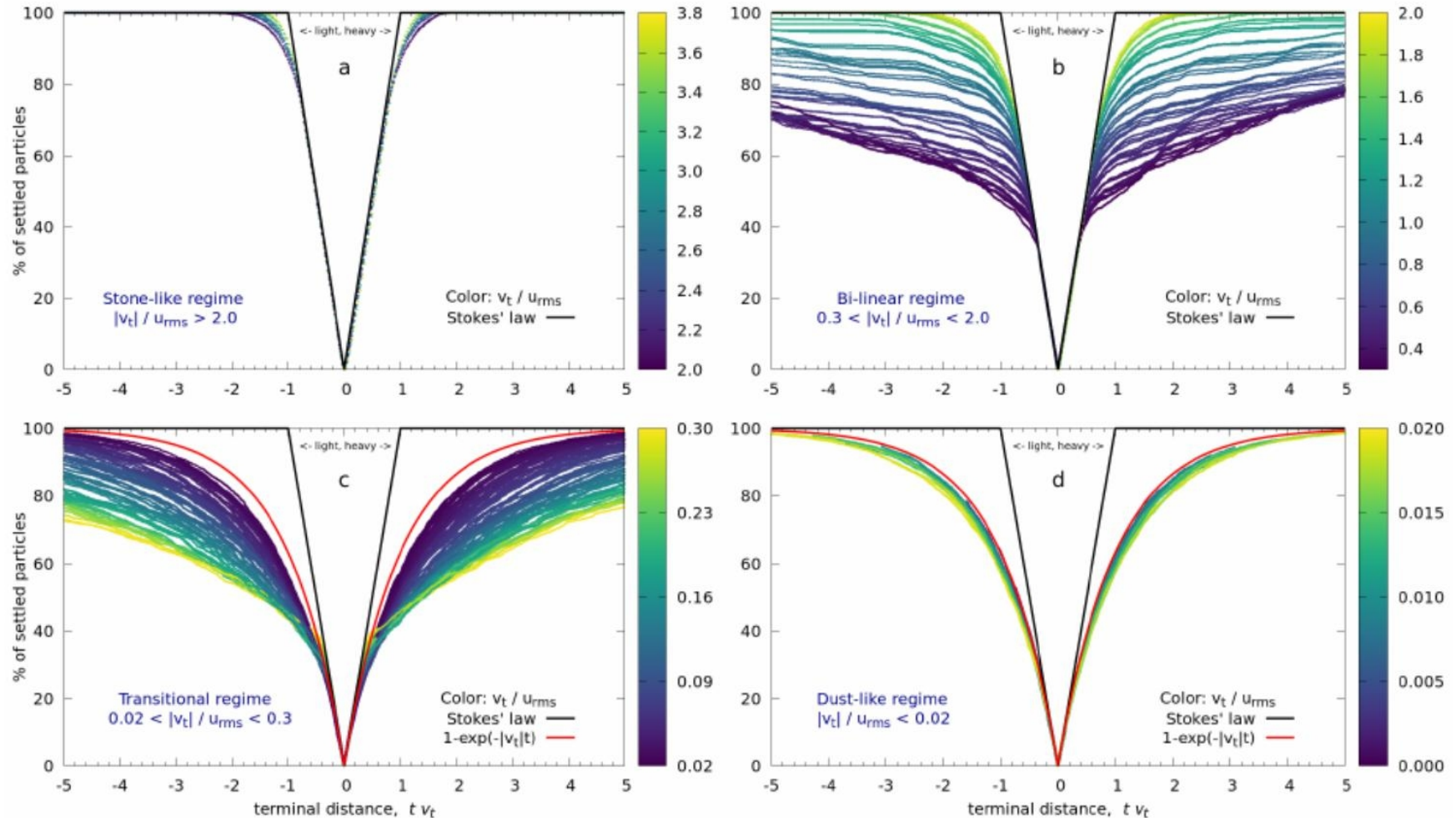
Particle model: Simplified Maxey-Riley equation (Basset-Boussinesq-Oseen model)

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \beta \frac{D\mathbf{u}}{Dt} + \frac{1}{St} (\mathbf{u} - \mathbf{v}) + \Lambda \hat{\mathbf{z}}, \\ \beta &= \frac{3\rho_f}{\rho_f + 2\rho_p}; \quad St = \frac{r_c^2 \sqrt{\alpha g \Delta T}}{3\nu \beta \sqrt{H}}; \quad \Lambda = \frac{\beta - 1}{\alpha \Delta T}.\end{aligned}$$

Example simulation



Results: settling curves



Our model

$$\text{M\&N(1989)} : \frac{dN}{dt} = -A v_t c(0), \quad c(0) = N/AH \quad \Rightarrow \quad N(t) = N_0 \exp\left(\frac{-v_t t}{H}\right)$$

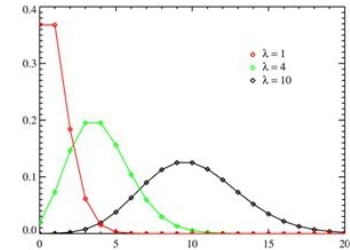
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$$dN = -\lambda N(t) dt$$

Poisson distribution

- Distribution for: how many light bulbs will burn out in a month?
 - Large building.
 - Maintenance department immediately replaces burned out bulb.
 - Choose λ = the average number that burn out in a month



(www.clipartof.com)

How many mathematicians does it take to change a light bulb?

In earlier work, Haynor[1] has shown that one mathematician can change a light bulb.

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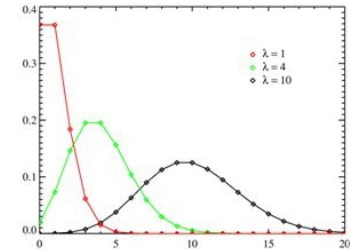
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$$dN = -(1 - \mathcal{P}_e) dN_c \stackrel{?}{=} -N v_t dt$$

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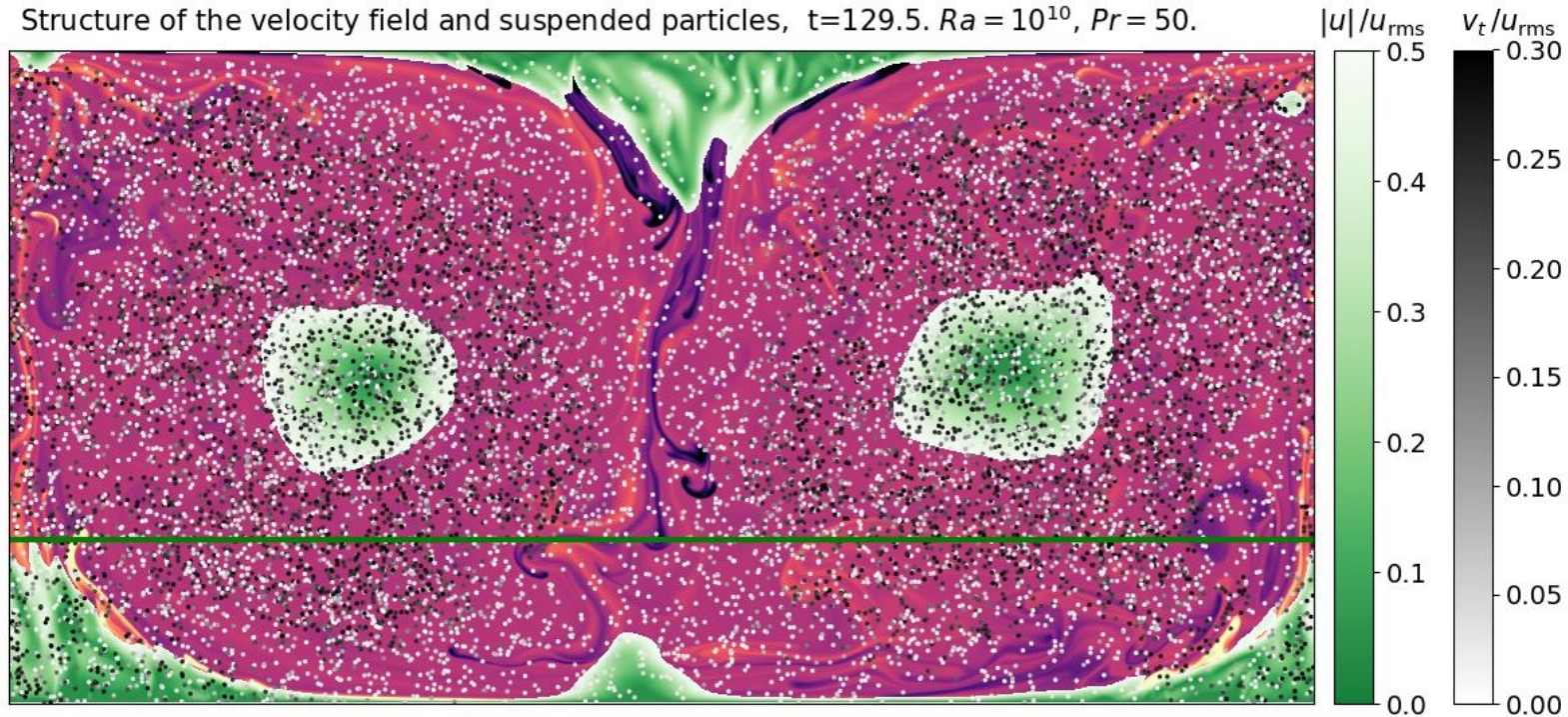
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Capture and escape

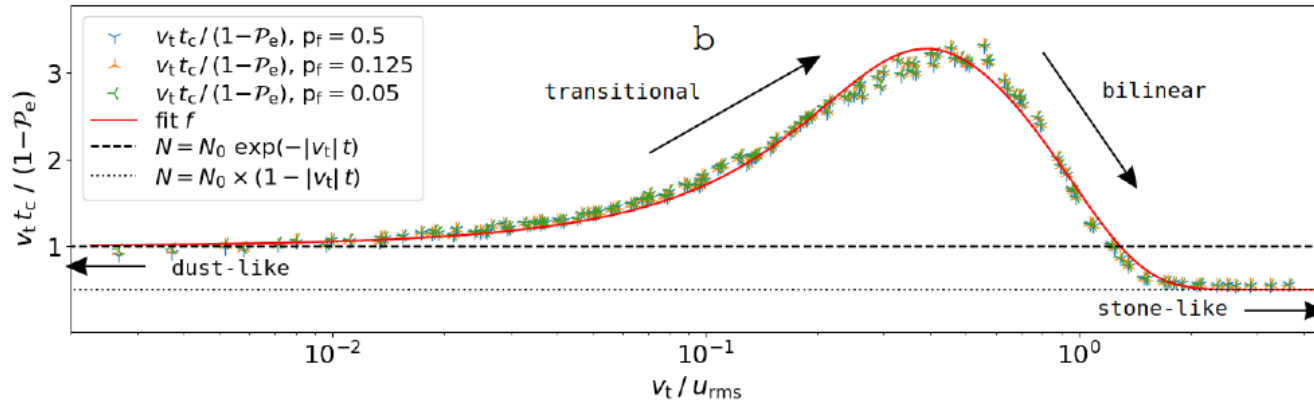
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Structure of the velocity field and suspended particles, $t=129.5$. $Ra = 10^{10}$, $Pr = 50$.



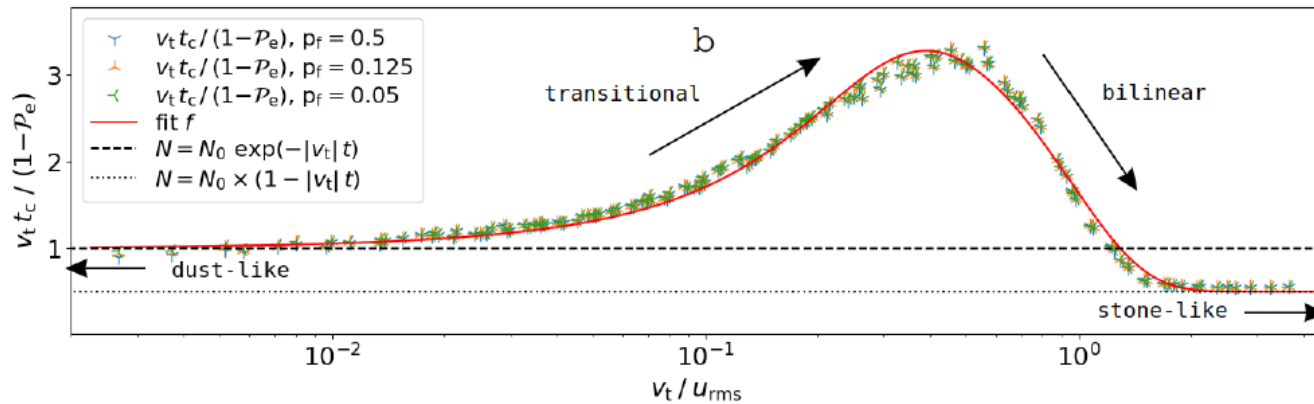
The distribution function

$$-dN = (1 - \mathcal{P}_e) dN_c = (1 - \mathcal{P}_e) \frac{N dt}{t_c} = \frac{(1 - \mathcal{P}_e)}{v_t t_c} N v_t dt$$



The distribution function

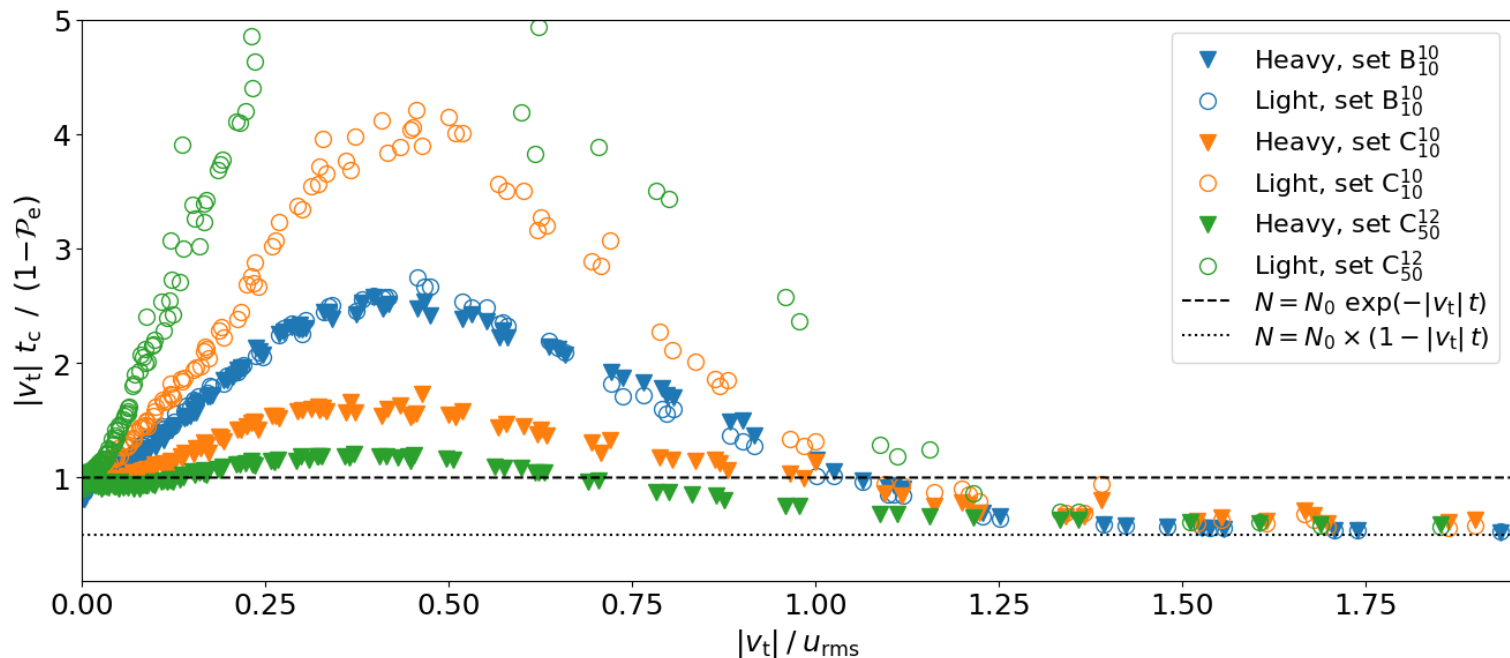
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$$\frac{t_c |v_t|}{1 - \mathcal{P}_e} = f(|v_t|/u_{\text{rms}}) := 0.5 + \frac{A}{\sigma\sqrt{2\pi}} \left[1 + \text{erf} \left(\frac{\lambda(x - \mu)}{\sigma\sqrt{2}} \right) \right] \exp \frac{-(x - \mu)^2}{2\sigma^2}$$

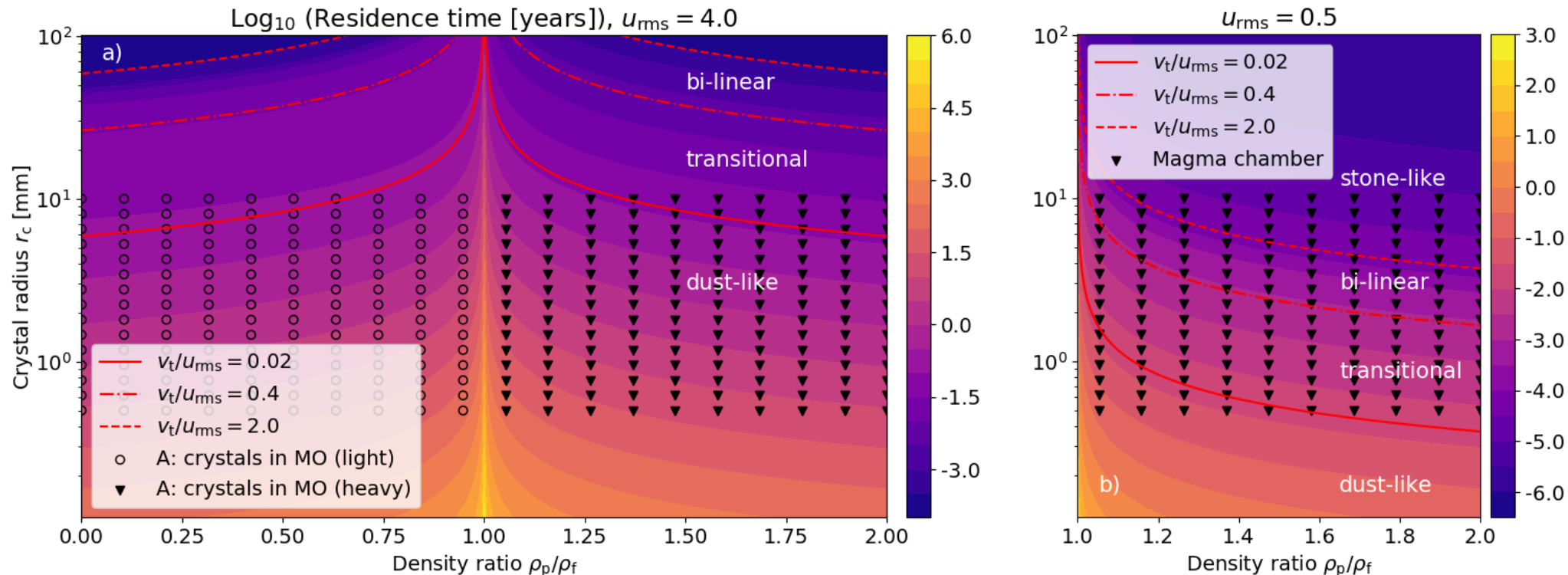
$$\frac{dN}{dt} = \frac{-|v_t| N}{f(|v_t|/u_{\text{rms}})} \Rightarrow N(t) = N_0 \exp \left(\frac{-|v_t| t}{f(|v_t|/u_{\text{rms}})} \right).$$

The influence of Ra, Pr, and β



Set	Max. m
Heavy, C_{50}^8	1.6
Light, C_{50}^8	1.7
Heavy, C_{10}^8	2.6
Light, C_{10}^8	2.9
Heavy, C_{50}^{10}	3.3
Light, C_{50}^{10}	4.3
Heavy, C_{10}^{10}	2.9
Light, C_{10}^{10}	5.0
Heavy, C_{10}^{12}	1.6
Light, C_{10}^{12}	4.0
Heavy, C_{50}^{12}	1.1
Light, C_{50}^{12}	9.6

Magma oceans and chambers: fractional crystallization



What next? Dark (blind?) alleys...

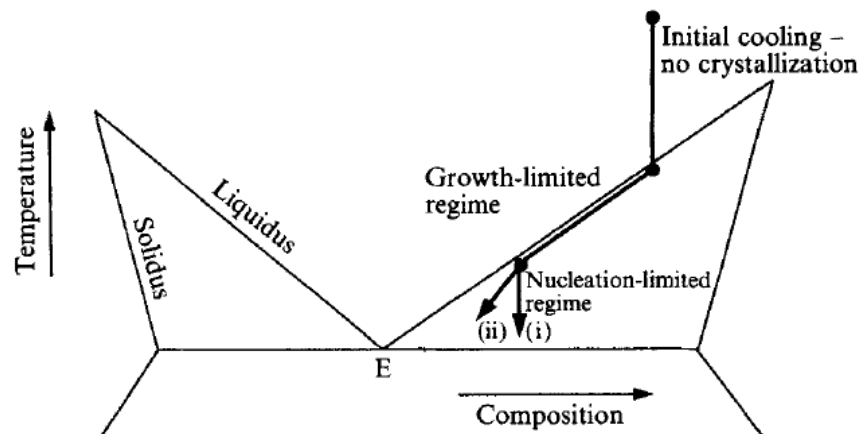
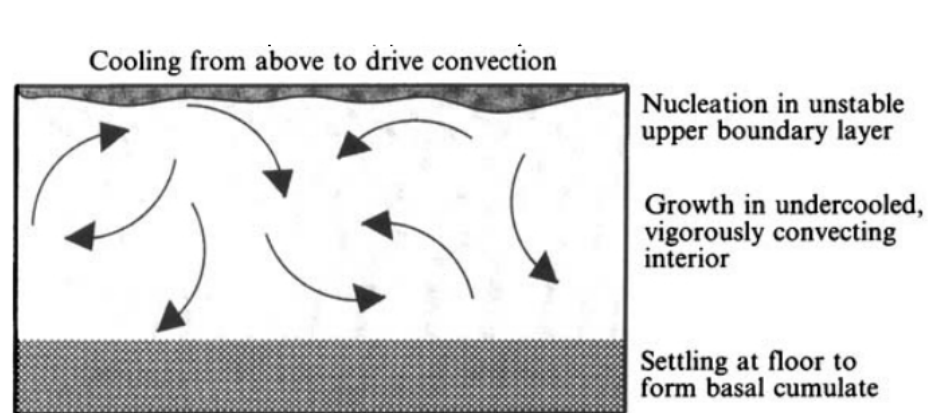
- Nucleation and growth of crystals
- Rotational flows
- Crystal feedback on the flow
- Re-entrainment

The nucleation, growth and settling of crystals from a turbulently convecting fluid

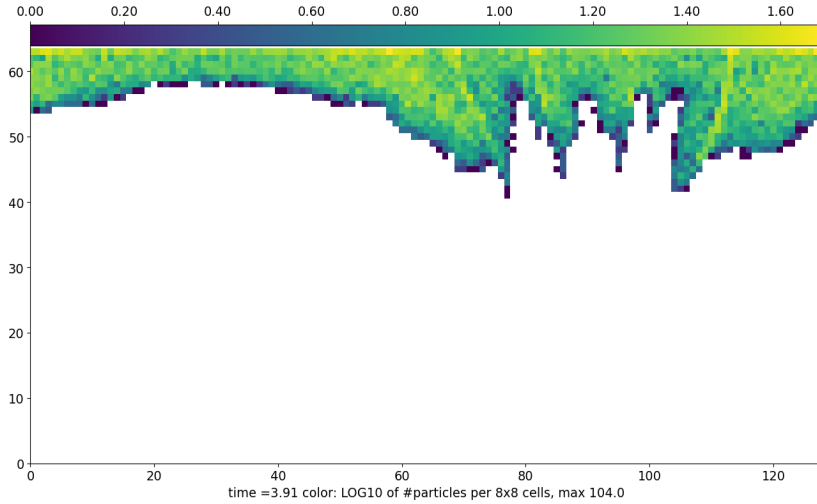
By RICHARD A. JARVIS^{1†} AND ANDREW W. WOODS²

¹Department of Geology and Geophysics, Yale University, New Haven, CT 06511, USA

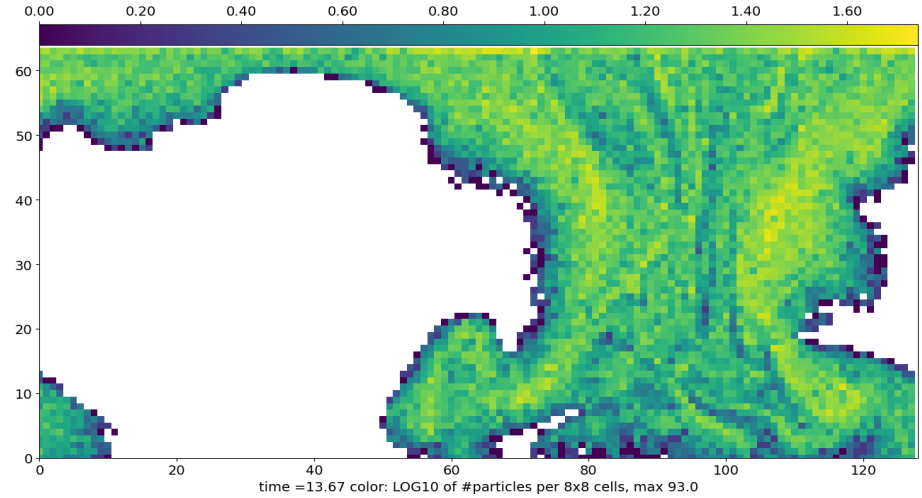
²Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK



Nucleation and growth



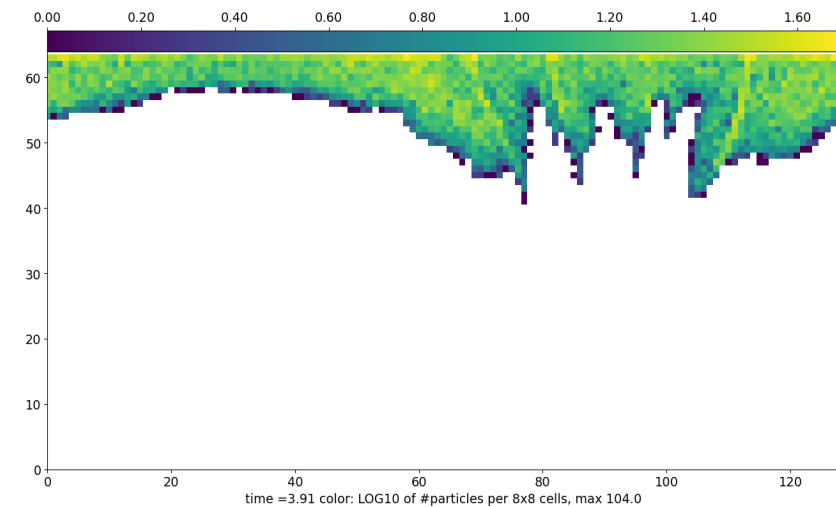
UC:
 $T_S - T$



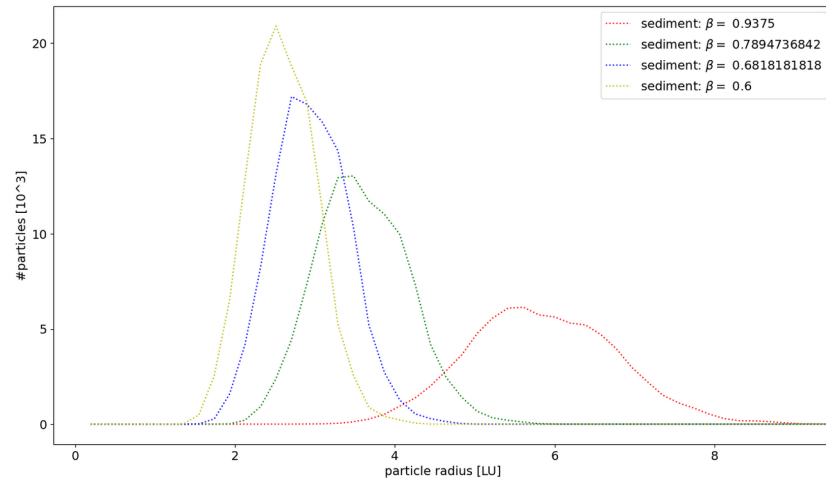
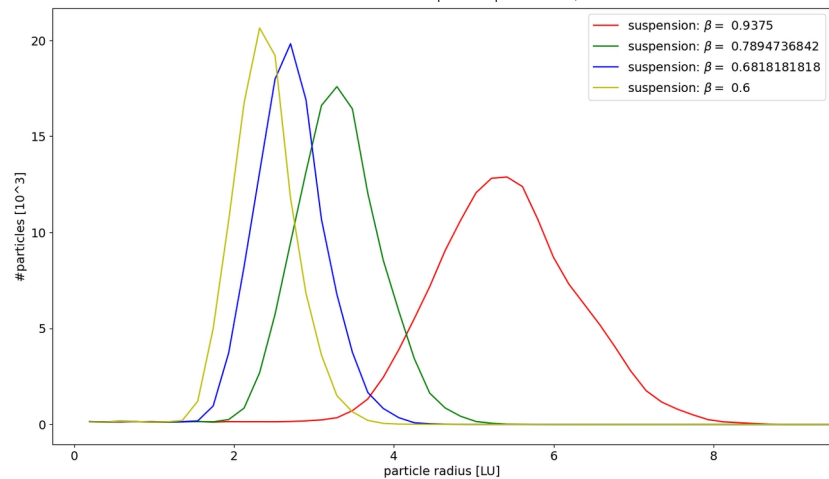
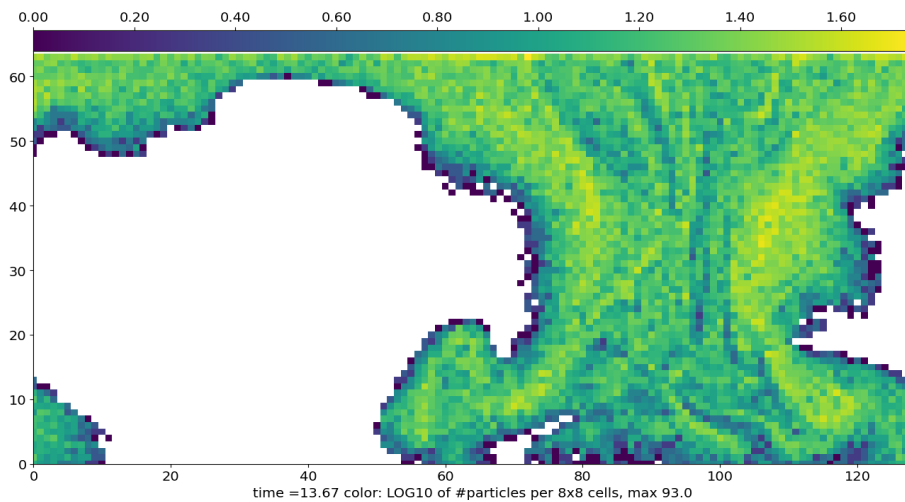
time



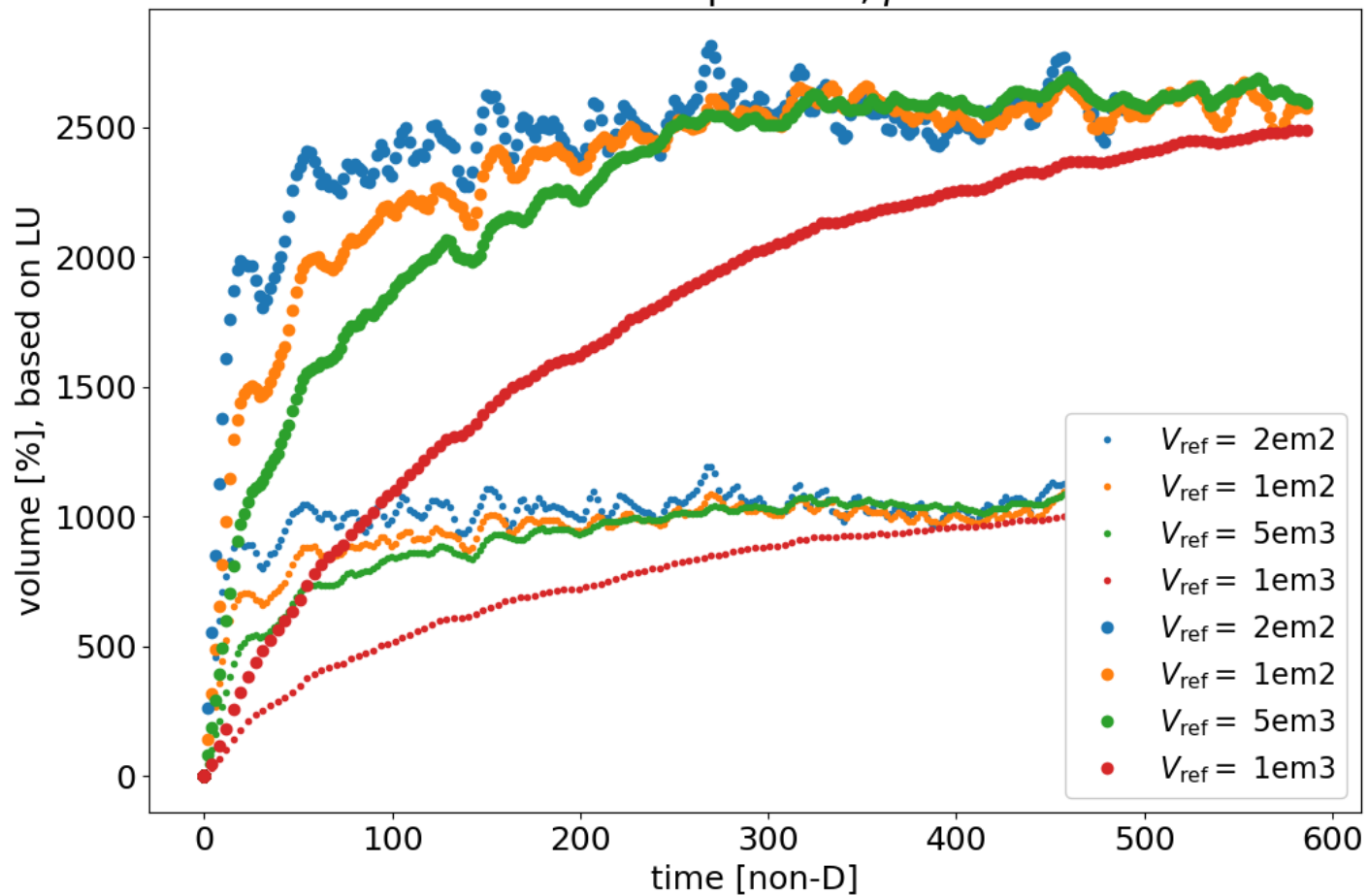
Nucleation and growth



UC:
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Volume of suspension, $\beta = 0.789$



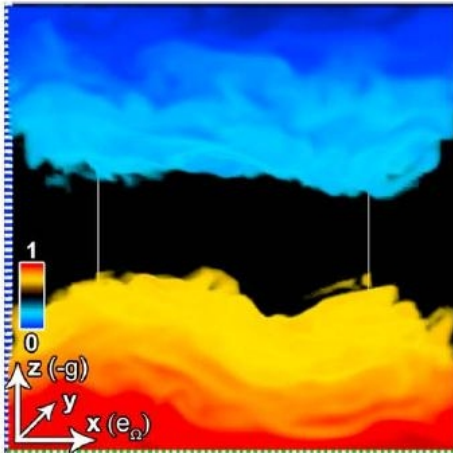
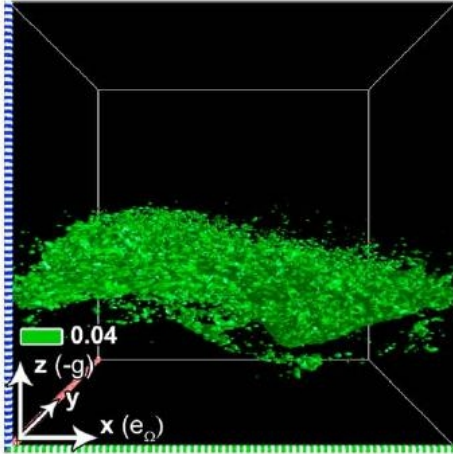
Rotational flows

Floating “ribbon”:

- balance between Coriolis force and gravity

Boundary conditions:

- free-slip, no-slip ?



(f) $Ro = 0.1$.

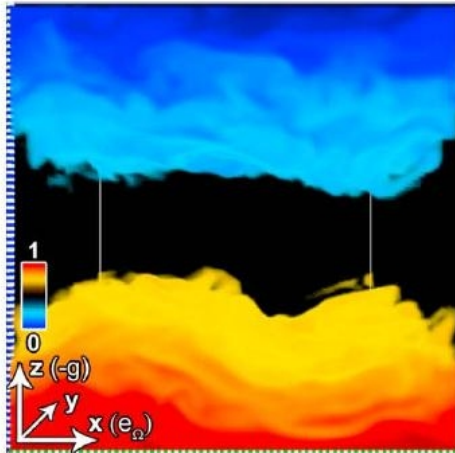
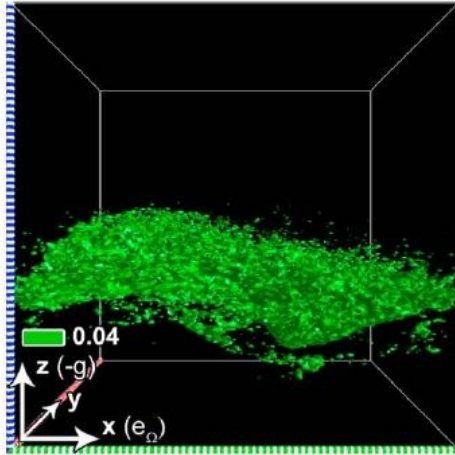
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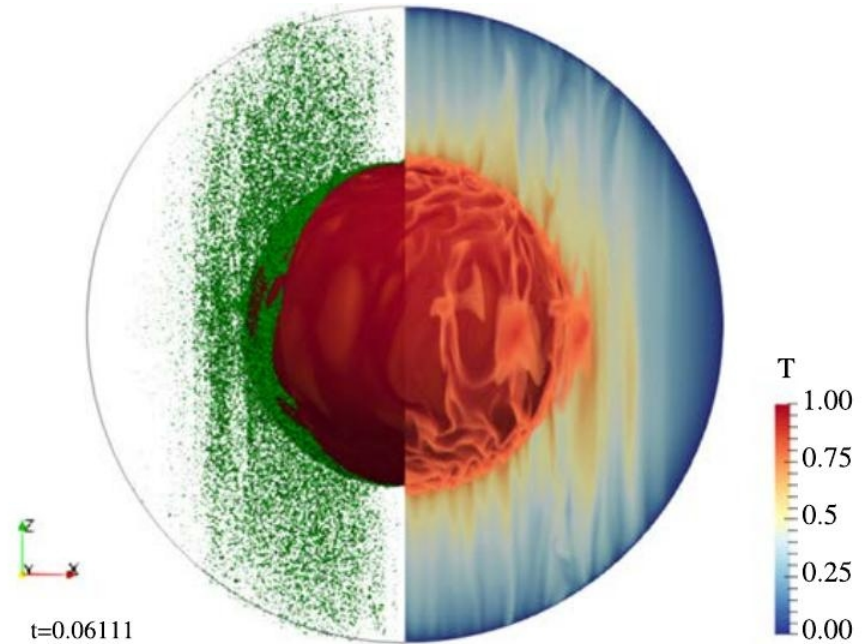
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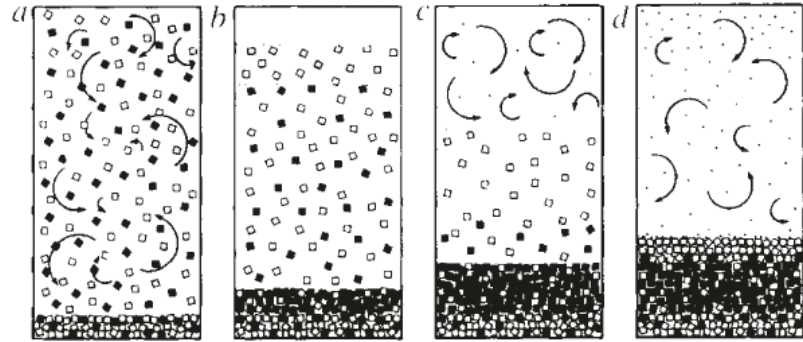
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(c) particle and temperature field snapshot.

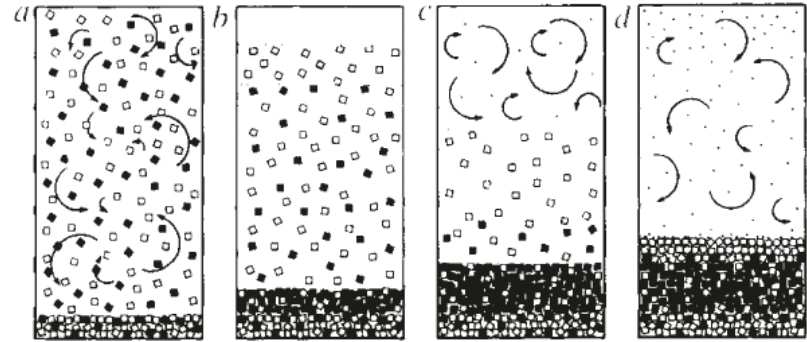
Crystal feedback on the flow

- Sedimentation events
(Koyaguchi et al, Sparks et al., Nature, 1990)

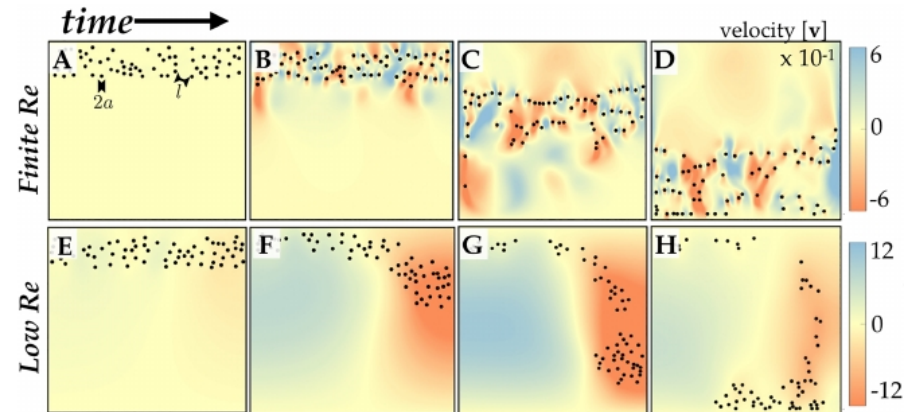


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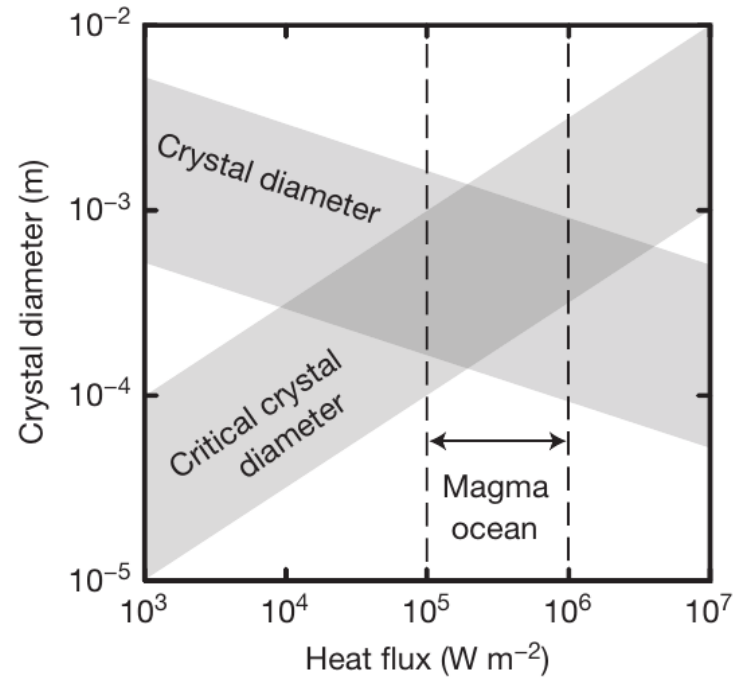
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- Crystal-driven convection
(Culha et al., JGR, 2020)

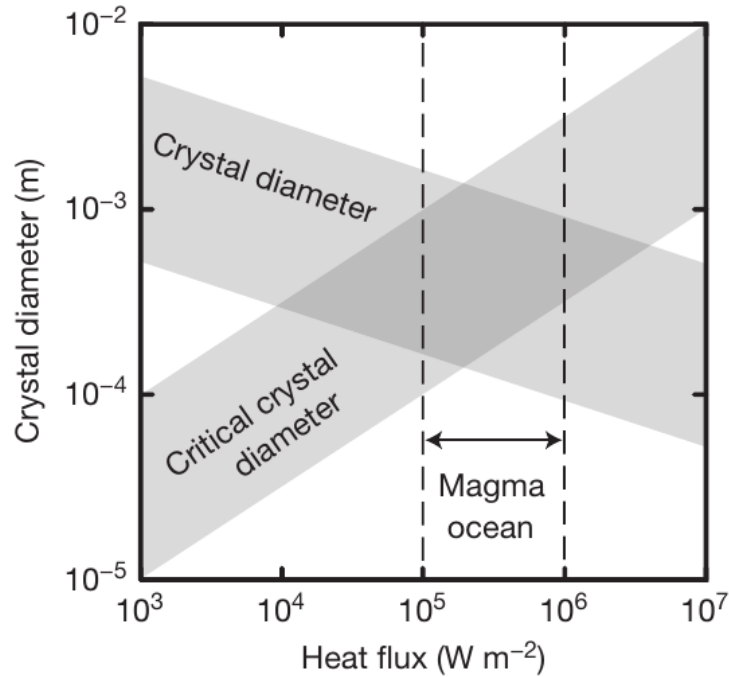


Re-entrainment

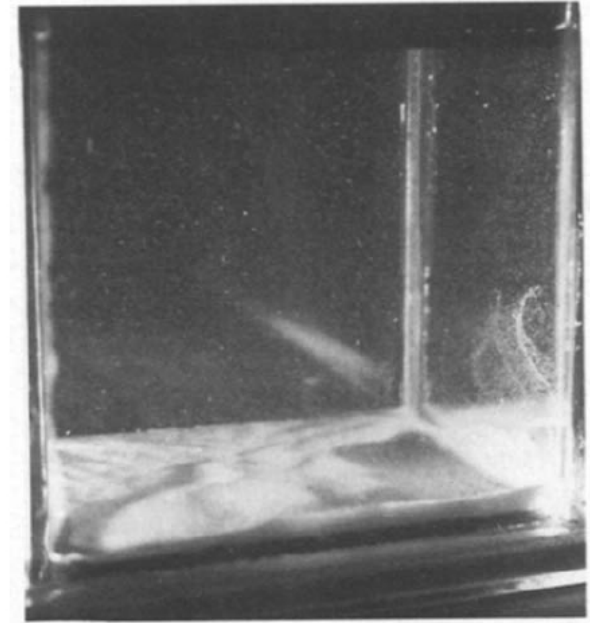
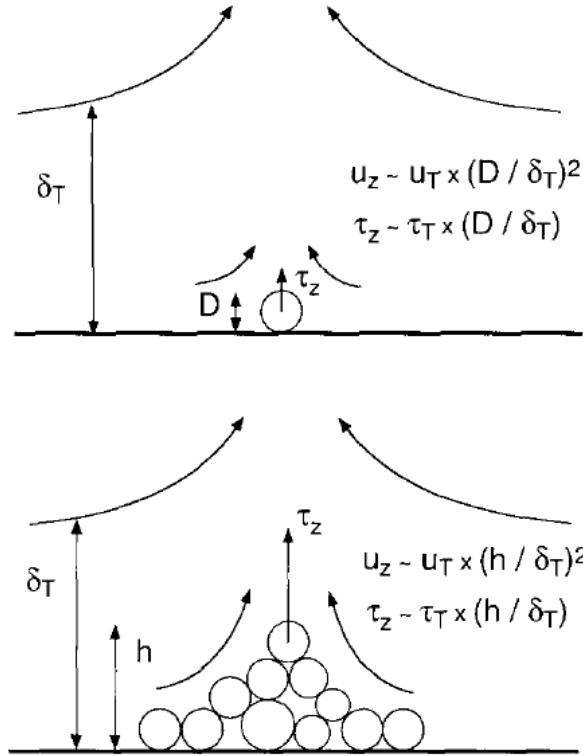


Solomatov, ToG 2015

Re-entrainment



Solomatov, ToG 2015



Series of papers by Solomatov, Stevenson, and Olson in 1993