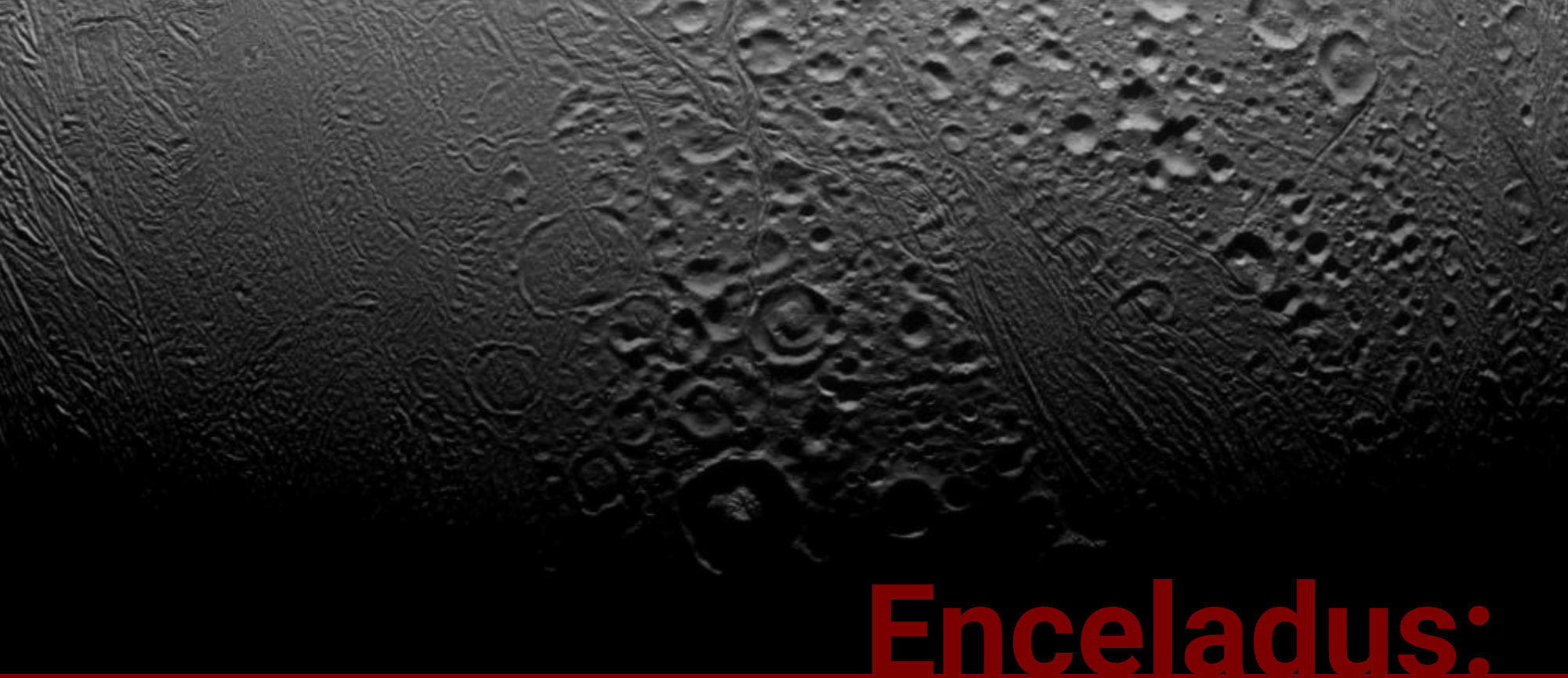
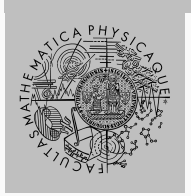


Enceladus:

Modelling convection with phase change: toward ice shell–ocean evolution on Enceladus

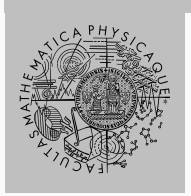
Geodynamics seminar 2021

Jiří Malík



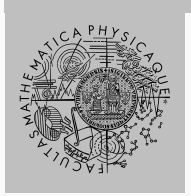
Outline

- Motivation



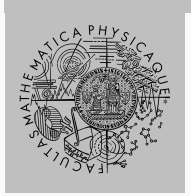
Outline

- Motivation
- Formulation



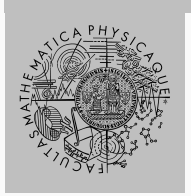
Outline

- Motivation
- Formulation
- (Purely) Conductive problem



Outline

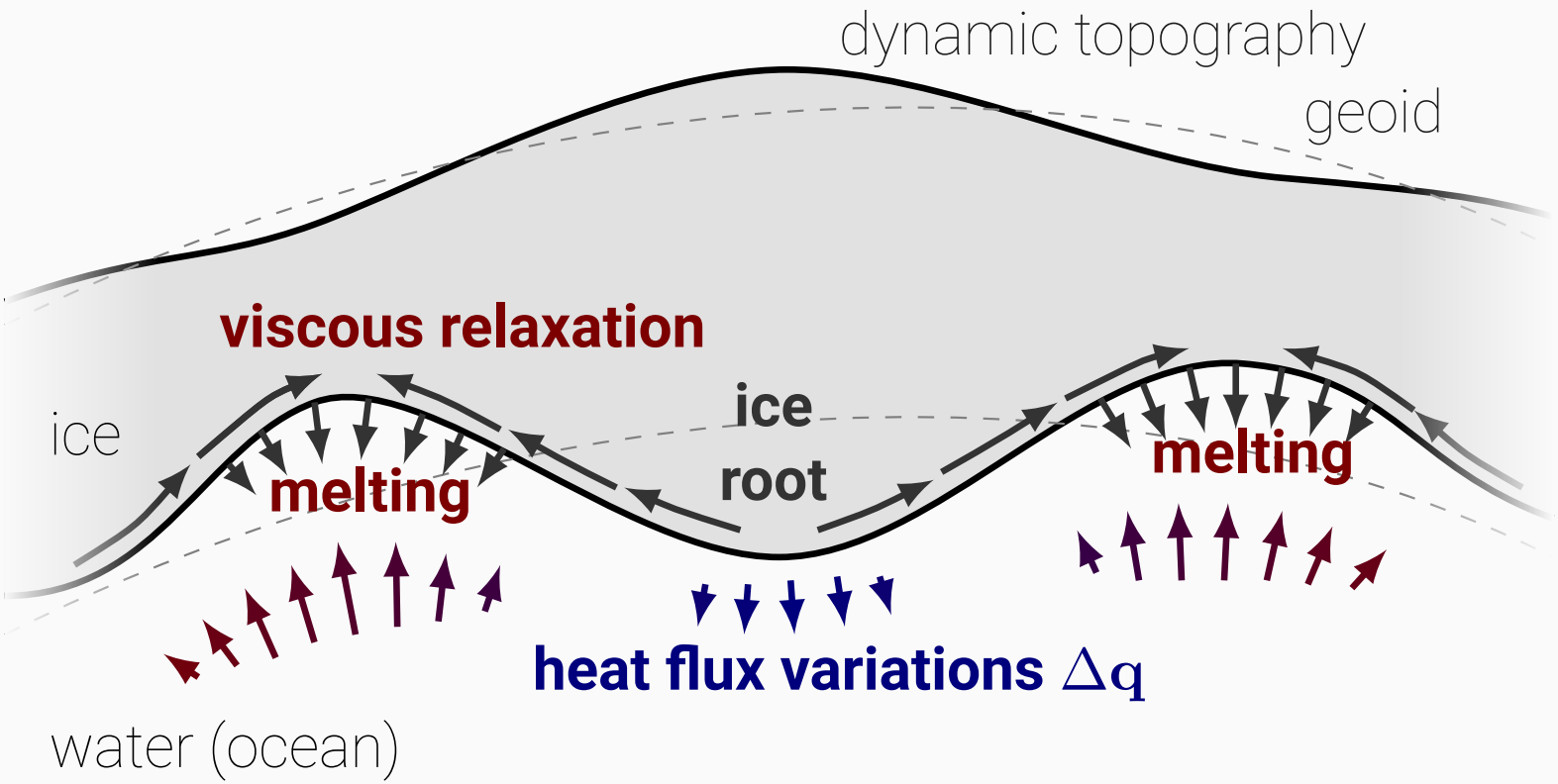
- Motivation
- Formulation
- (Purely) Conductive problem
- Convective problem

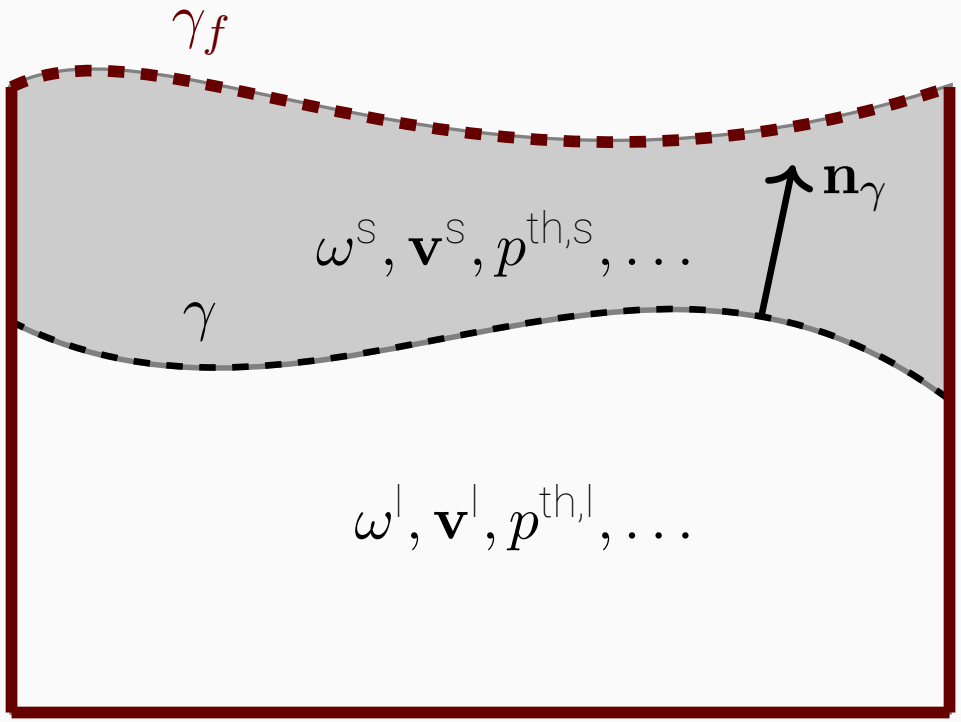


Outline

- Motivation
- Formulation
- (Purely) Conductive problem
- Convective problem
- Application

Interaction of ice crust and subsurface ocean




γ_f
$\mathbf{n}_\gamma$
$\omega^{\mathsf{s}}, \mathbf{v}^{\mathsf{s}}, p^{\mathsf{th,s}}, \ldots$
γ
$\omega^{\mathsf{l}}, \mathbf{v}^{\mathsf{l}}, p^{\mathsf{th,l}}, \ldots$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}} \mathbb{I} + \mathbb{S}$$

$$\rho \dot{e} = -\operatorname{div} \mathbf{q} + \mathbb{T} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho \mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T} \mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

$$[\![\rho\big(1/2|\mathbf{v}|^2 + e\big)(\mathbf{w} - \mathbf{v}) + \mathbf{v}\mathbb{T} - \mathbf{q}]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}} \mathbb{I} + \mathbb{S}$$

$$\rho \dot{h} = -\operatorname{div} \mathbf{q} + \overline{p^{\mathrm{th}}}^{\cdot} + \mathbb{S} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho \mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T} \mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

$$\rho^{\mathrm{s/l}} \left(\mathbf{v}^{\mathrm{s/l}} - \mathbf{w} \right) \cdot \mathbf{n}_\gamma [\![h + 1/2 |\mathbf{v}|^2]\!] = [\![-\mathbf{q} + \mathbb{S}\mathbf{v}]\!] \cdot \mathbf{n}_\gamma \qquad \text{on } \gamma$$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}} \mathbb{I} + \mathbb{S}$$

$$\rho \dot{h} = -\operatorname{div} \mathbf{q} + \overline{p^{\mathrm{th}}}^{\bullet} + \mathbb{S} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho \mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T} \mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

$$\mathcal{M}[\![h + 1/2|\mathbf{v}|^2]\!] = [\![-\mathbf{q} + \mathbb{S}\mathbf{v}]\!] \cdot \mathbf{n}_\gamma \qquad \text{on } \gamma$$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}} \mathbb{I} + \mathbb{S}$$

$$\rho \dot{h} = -\operatorname{div} \mathbf{q} + \overline{p^{\mathrm{th}}}^{\bullet} + \mathbb{S} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho \mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T} \mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

$$\mathcal{M}[\![h]\!] = [\![-\mathbf{q} + \mathbb{S}\mathbf{v}]\!] \cdot \mathbf{n}_\gamma - \mathcal{M}[\![1/2|\mathbf{v}|^2]\!] \qquad \text{on } \gamma$$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}} \mathbb{I} + \mathbb{S}$$

$$\rho \dot{h} = -\operatorname{div} \mathbf{q} + \overline{p^{\mathrm{th}}}^{\bullet} + \mathbb{S} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho \mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T}\mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

$$\mathcal{M}\mathrm{L}_{\mathrm{m}} = [\![-\mathbf{q} + \mathbb{S}\mathbf{v}]\!] \cdot \mathbf{n}_\gamma - \mathcal{M}[\![1/2|\mathbf{v}|^2]\!] \qquad \text{on } \gamma$$

Conservation laws in local form:

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho\dot{\mathbf{v}} = \operatorname{div} \mathbb{T} + \mathbf{b} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\mathbb{T} = -p^{\mathrm{th}}\mathbb{I} + \mathbb{S}$$

$$\rho\dot{h} = -\operatorname{div} \mathbf{q} + \overline{\dot{p^{\mathrm{th}}}} + \mathbb{S} : \mathbb{D} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Interface conditions:

$$[\![\rho(\mathbf{w} - \mathbf{v})]\!] \cdot \mathbf{n}_\gamma = 0 \qquad \text{on } \gamma$$

$$[\![\rho\mathbf{v}((\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}_\gamma)]\!] + [\![\mathbb{T}\mathbf{n}_\gamma]\!] = \mathbf{0} \qquad \text{on } \gamma$$

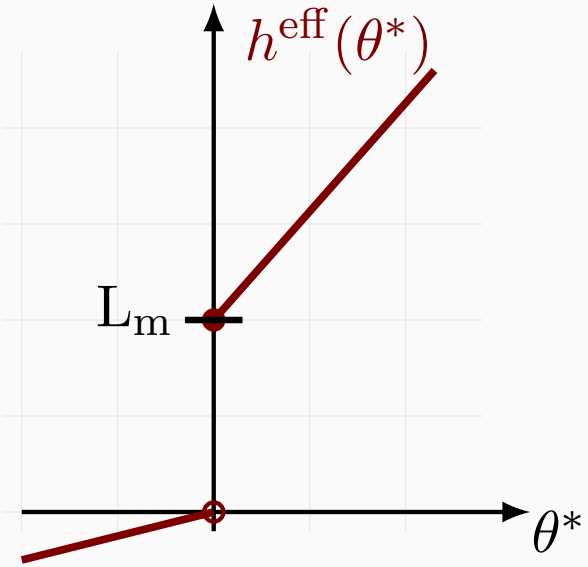
$$\mathcal{M}\mathrm{L}_{\mathrm{m}} = [\![-\mathbf{q} + \mathbb{S}\mathbf{v}]\!] \cdot \mathbf{n}_\gamma - \mathcal{M}[\![1/2|\mathbf{v}|^2]\!] \qquad \text{on } \gamma$$

$$\rho \dot{h} = -\operatorname{div} \mathbf{q} + \overline{p^{\mathrm{th}}}^{\bullet} + \mathbb{S} : \mathbb{D} \quad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho \frac{\partial h}{\partial t} = -\operatorname{div} \mathbf{q} + \overline{\dot{p}^{\mathrm{th}}} + \mathbb{S} : \mathbb{D} \quad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

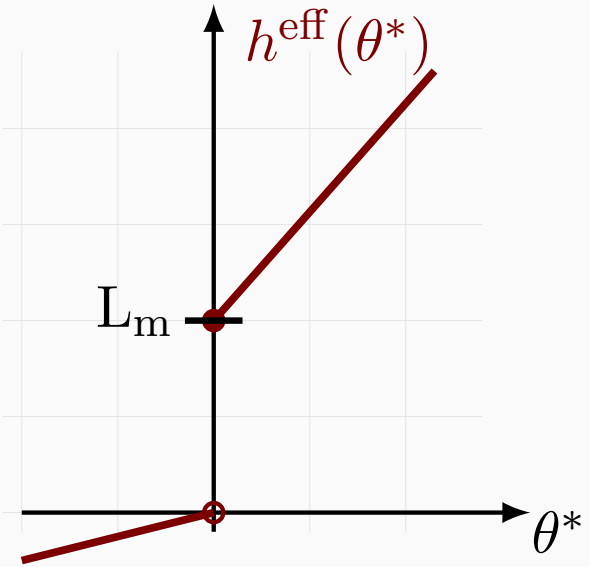
$$\rho \frac{\partial h}{\partial t} = -\operatorname{div} \mathbf{q} \quad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$h^{\mathrm{eff}}(\theta) \stackrel{\mathrm{def}}{=} \begin{cases} h_0 + c^{\mathrm{s}}(\theta - \theta_{\mathrm{m}}), & \theta < \theta_{\mathrm{m}}, \\ h_0 + \mathrm{L}_{\mathrm{m}} + c^{\mathrm{l}}(\theta - \theta_{\mathrm{m}}), & \theta \geq \theta_{\mathrm{m}}, \end{cases}$$



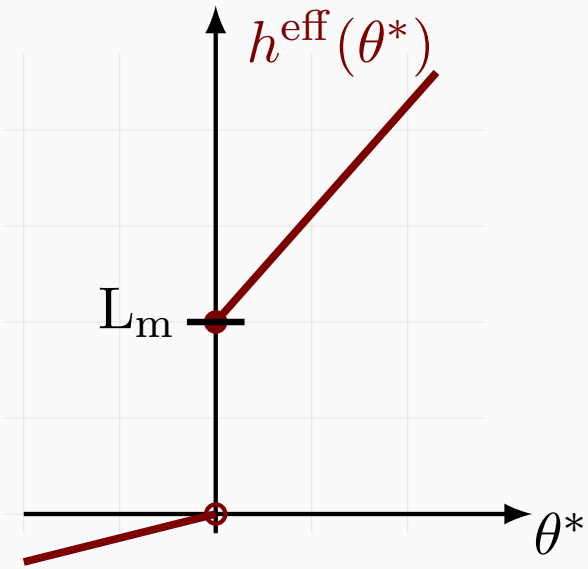
$$\rho \frac{\partial h^{\mathrm{eff}}}{\partial t} = -\operatorname{div} \mathbf{q} \quad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

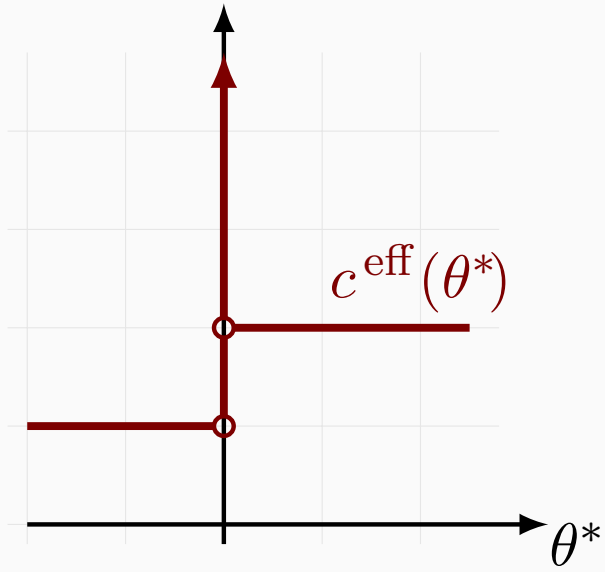
$$h^{\mathrm{eff}}(\theta) \stackrel{\mathrm{def}}{=} \begin{cases} h_0 + c^{\mathrm{s}}(\theta - \theta_{\mathrm{m}}), & \theta < \theta_{\mathrm{m}}, \\ h_0 + \mathrm{L}_{\mathrm{m}} + c^{\mathrm{l}}(\theta - \theta_{\mathrm{m}}), & \theta \geq \theta_{\mathrm{m}}, \end{cases}$$



$$\rho\, c^{\,\mathrm{eff}}\,\frac{\partial\theta}{\partial t} = -\operatorname{div}\mathbf{q} \quad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

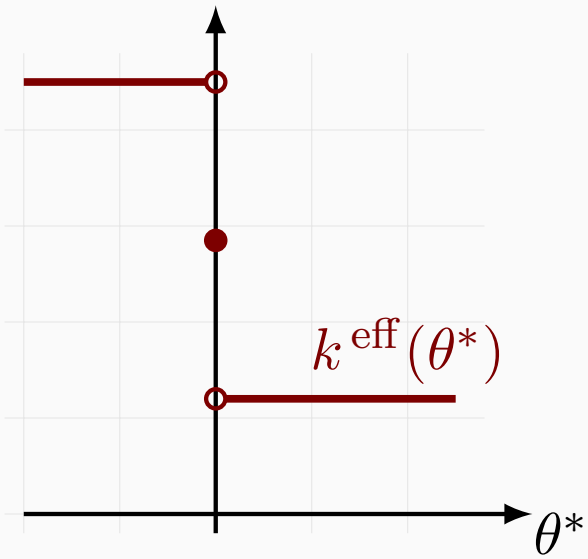
$$h^{\mathrm{eff}}(\theta) \stackrel{\mathrm{def}}{=} \begin{cases} h_0 + c^{\mathrm{s}}(\theta - \theta_{\mathrm{m}})\,, & \theta < \theta_{\mathrm{m}}\,, \\ h_0 + \mathrm{L}_{\mathrm{m}} + c^{\mathrm{l}}(\theta - \theta_{\mathrm{m}})\,, & \theta \geq \theta_{\mathrm{m}}\,, \end{cases}$$

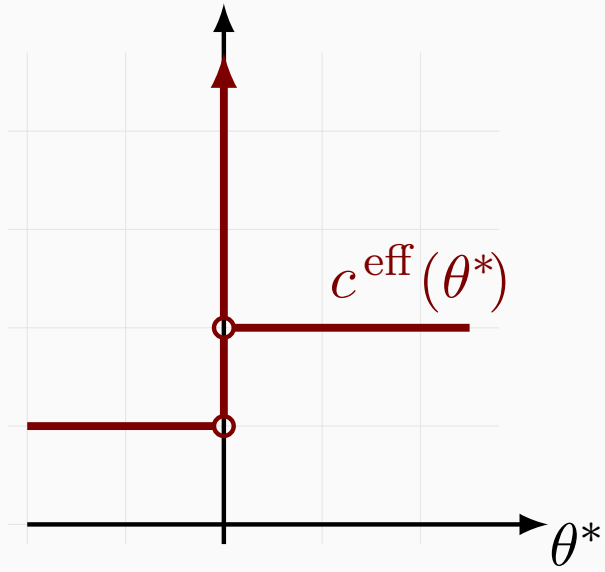




$$\rho\, c^{\,\mathrm{eff}}\frac{\partial\theta}{\partial t} = \operatorname{div}\left(k^{\,\mathrm{eff}}\nabla\theta\right) \quad \text{in } \omega^{\mathrm{l}}\cup\omega^{\mathrm{s}}$$

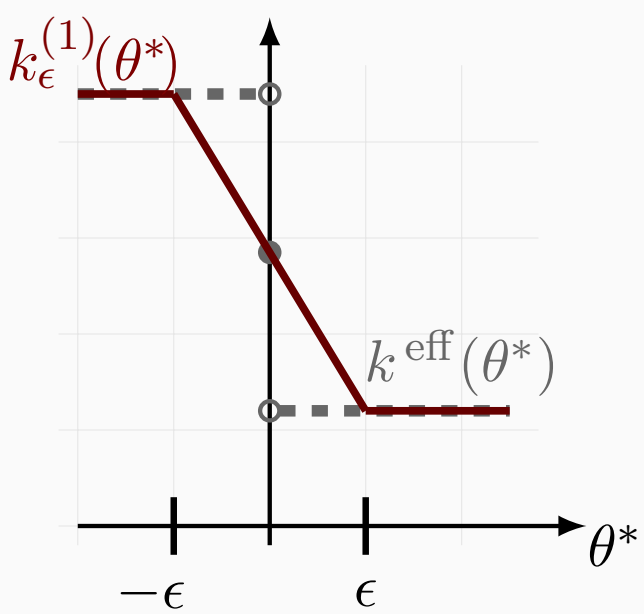
$$h^{\mathrm{eff}}(\theta) \stackrel{\mathrm{def}}{=} \begin{cases} h_0 + c^{\mathrm{s}}(\theta-\theta_{\mathrm{m}})\,, & \theta < \theta_{\mathrm{m}}\,, \\ h_0 + \mathrm{L}_{\mathrm{m}} + c^{\mathrm{l}}(\theta-\theta_{\mathrm{m}})\,, & \theta \geq \theta_{\mathrm{m}}\,, \end{cases}$$

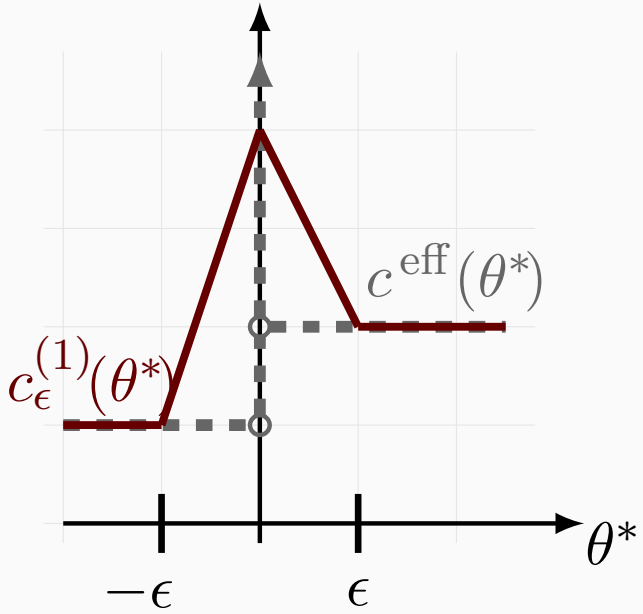


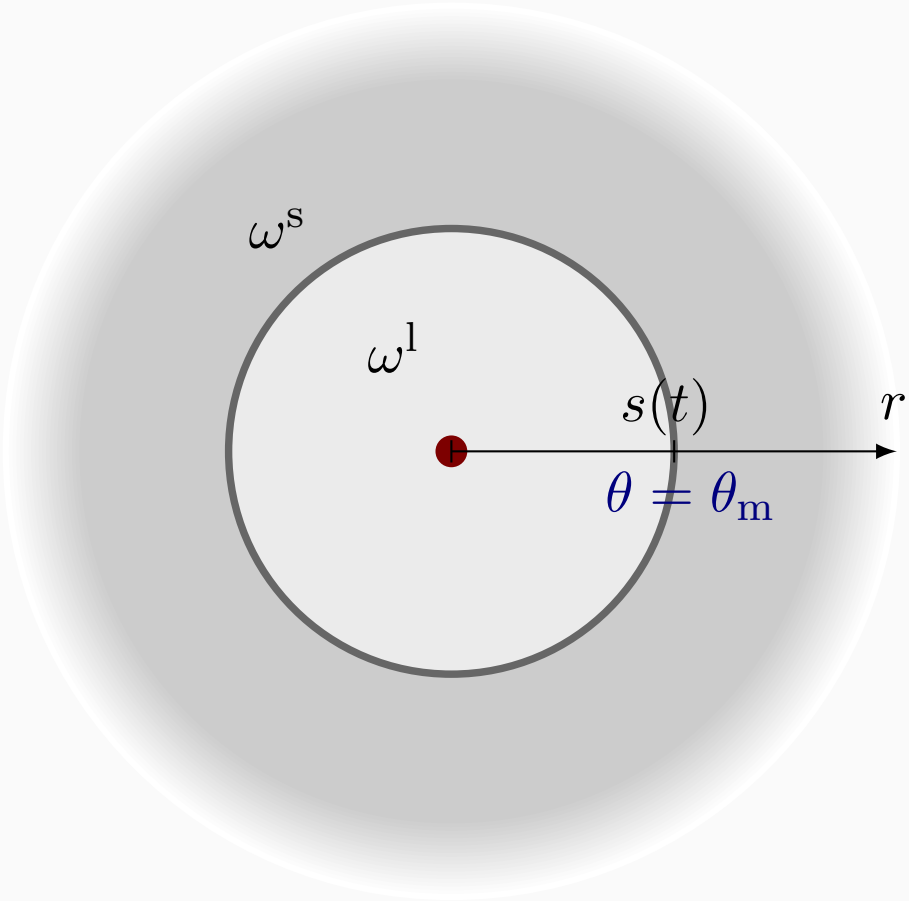


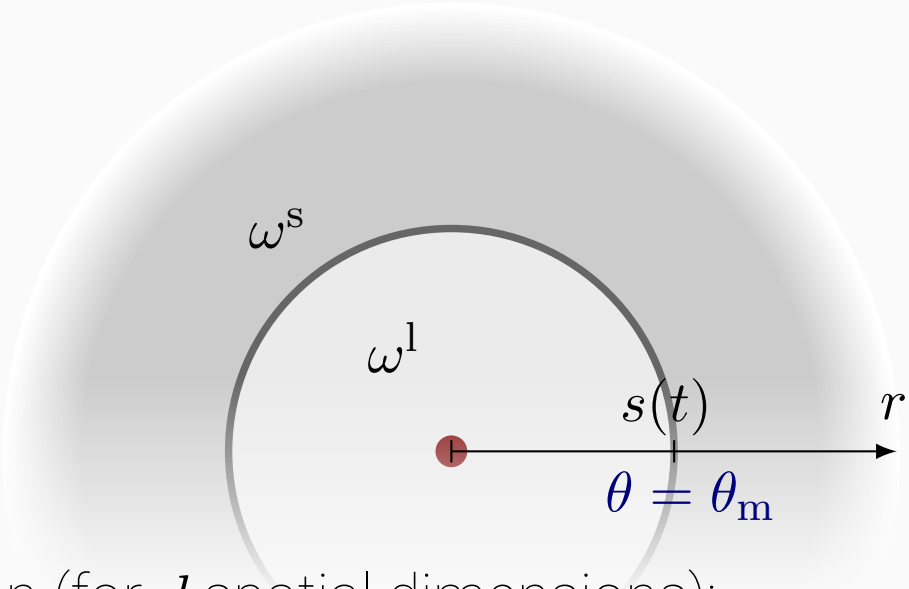
$$\rho\, c_\epsilon^{(1)} \frac{\partial \theta}{\partial t} = \operatorname{div}\left(k_\epsilon^{(1)} \nabla \theta\right) \quad \text{in } \omega$$

$$h^{\mathrm{eff}}(\theta) \stackrel{\mathrm{def}}{=} \begin{cases} h_0 + c^{\mathrm{s}}(\theta - \theta_{\mathrm{m}})\,, & \theta < \theta_{\mathrm{m}}\,, \\ h_0 + \mathrm{L}_{\mathrm{m}} + c^{\mathrm{l}}(\theta - \theta_{\mathrm{m}})\,, & \theta \geq \theta_{\mathrm{m}}\,, \end{cases}$$






ω^{s}
ω^{l}
$s(t)$
r
$\theta = \theta_{\mathrm{m}}$

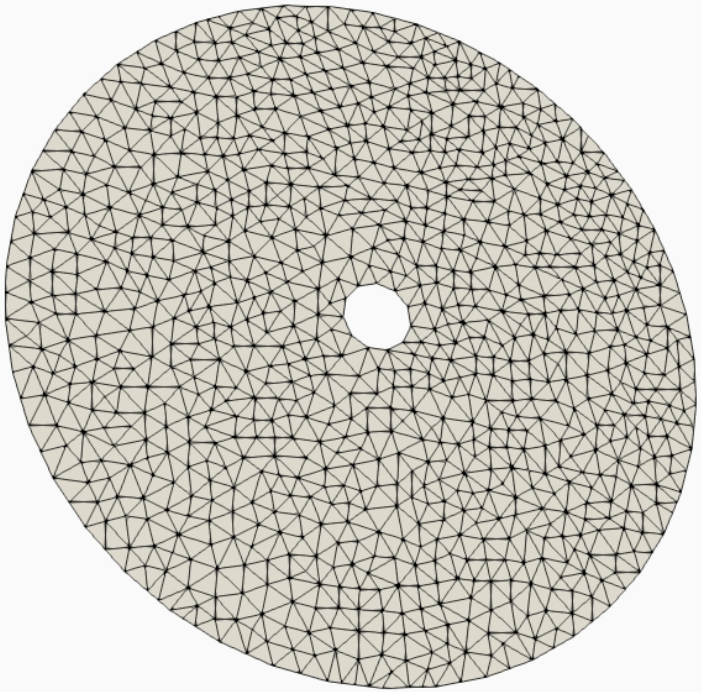


General solution (for d spatial dimensions):

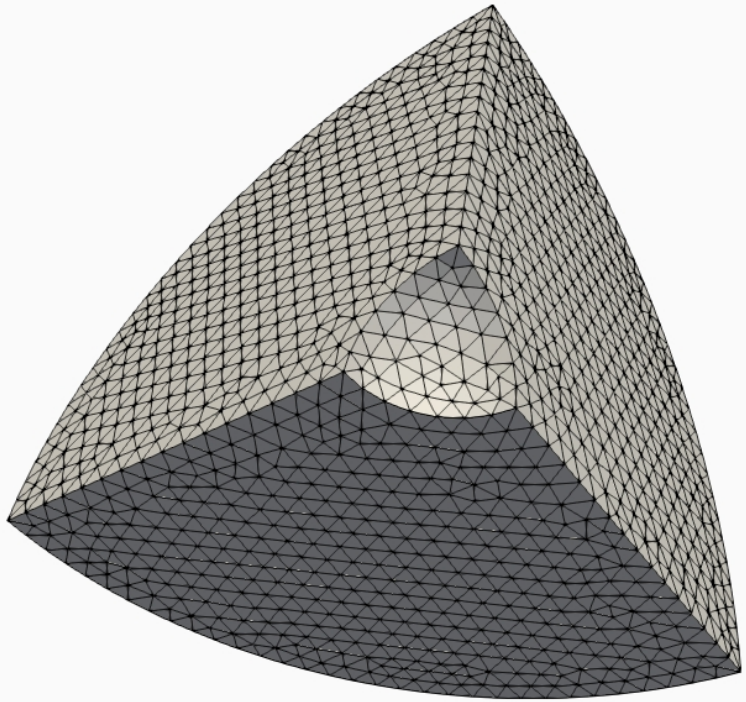
$$\theta_d^{\mathrm{l}}(t,r) = C_{1,d}^{\mathrm{l}}\Gamma\left(1-\frac{d}{2},\frac{r^2}{4\alpha^{\mathrm{l}}t}\right) + C_{2,d}^{\mathrm{l}}\,,\quad r\in(0,s(t))\,,$$

$$\theta_d^{\mathrm{s}}(t,r) = C_{1,d}^{\mathrm{s}}\Gamma\left(1-\frac{d}{2},\frac{r^2}{4\alpha^{\mathrm{s}}t}\right) + C_{2,d}^{\mathrm{s}}\,,\quad r\in(s(t),+\infty)\,,$$

$$s(t) = 2\lambda_d\sqrt{t}\,,\qquad\qquad t\in(0,+\infty)$$



(a) $2d$ mesh (**mshr**)



(b) $3d$ mesh (**gmsh**)

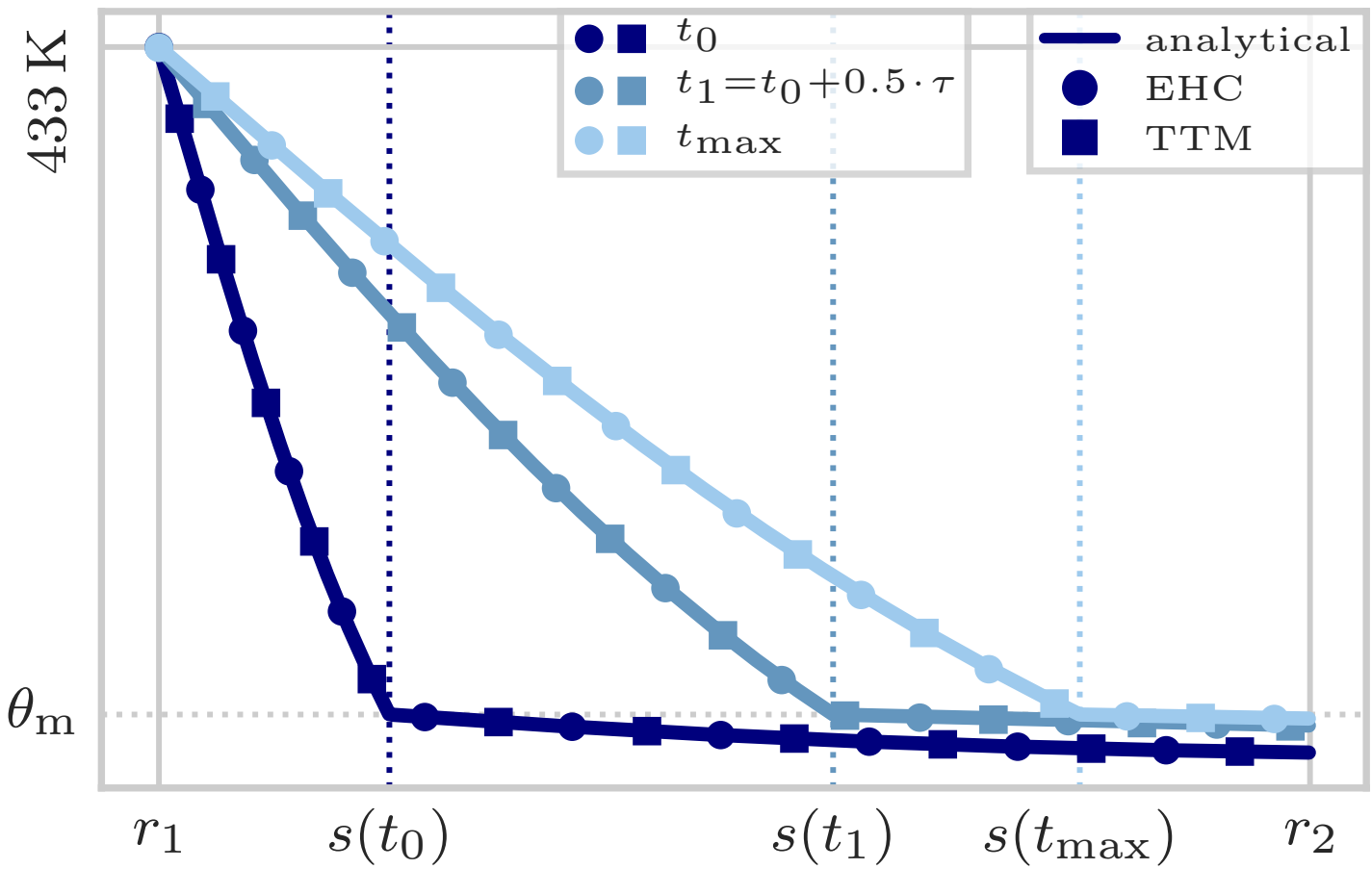


Figure: Temperature distribution $1d$

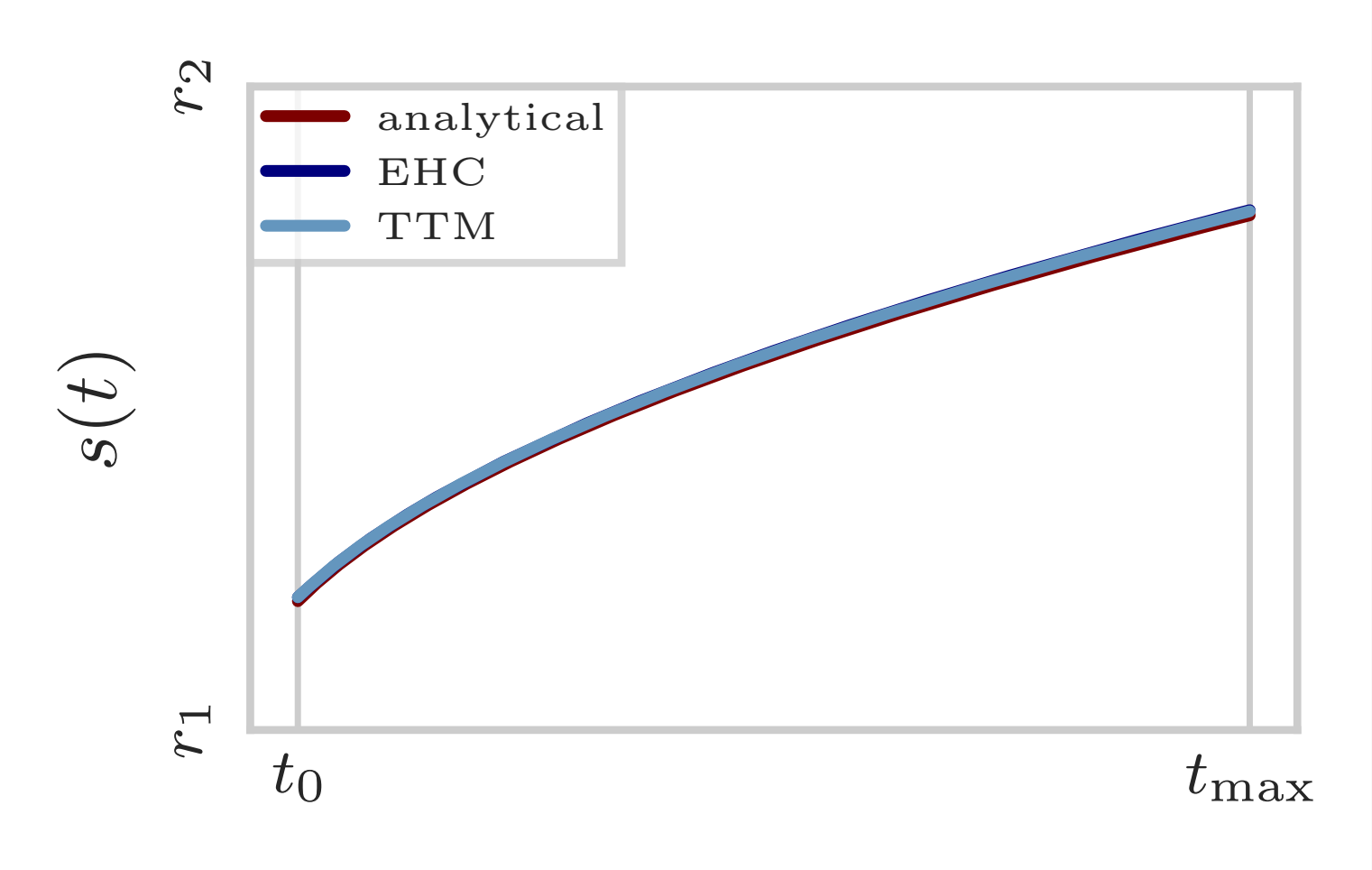


Figure: Position of the melting front

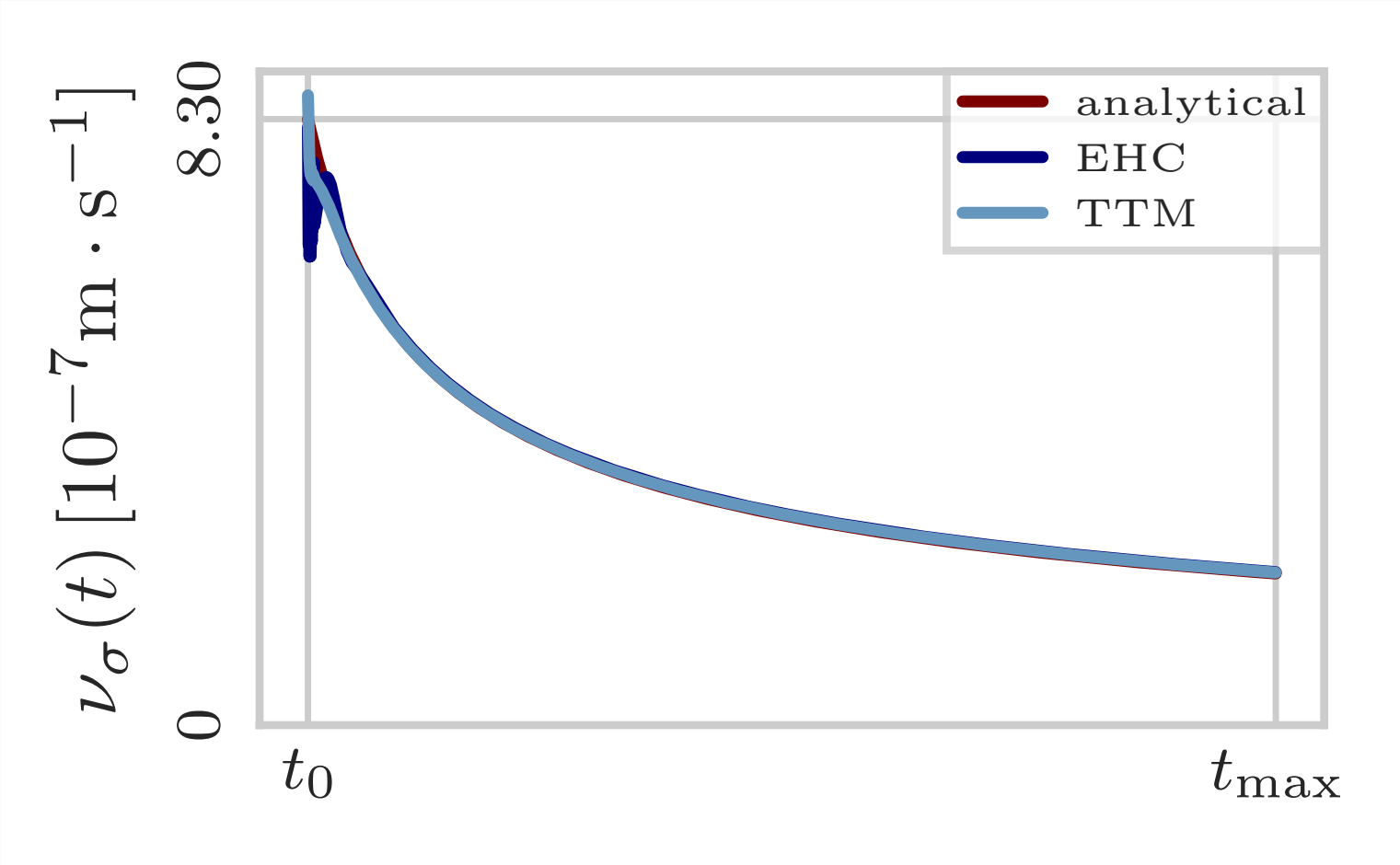
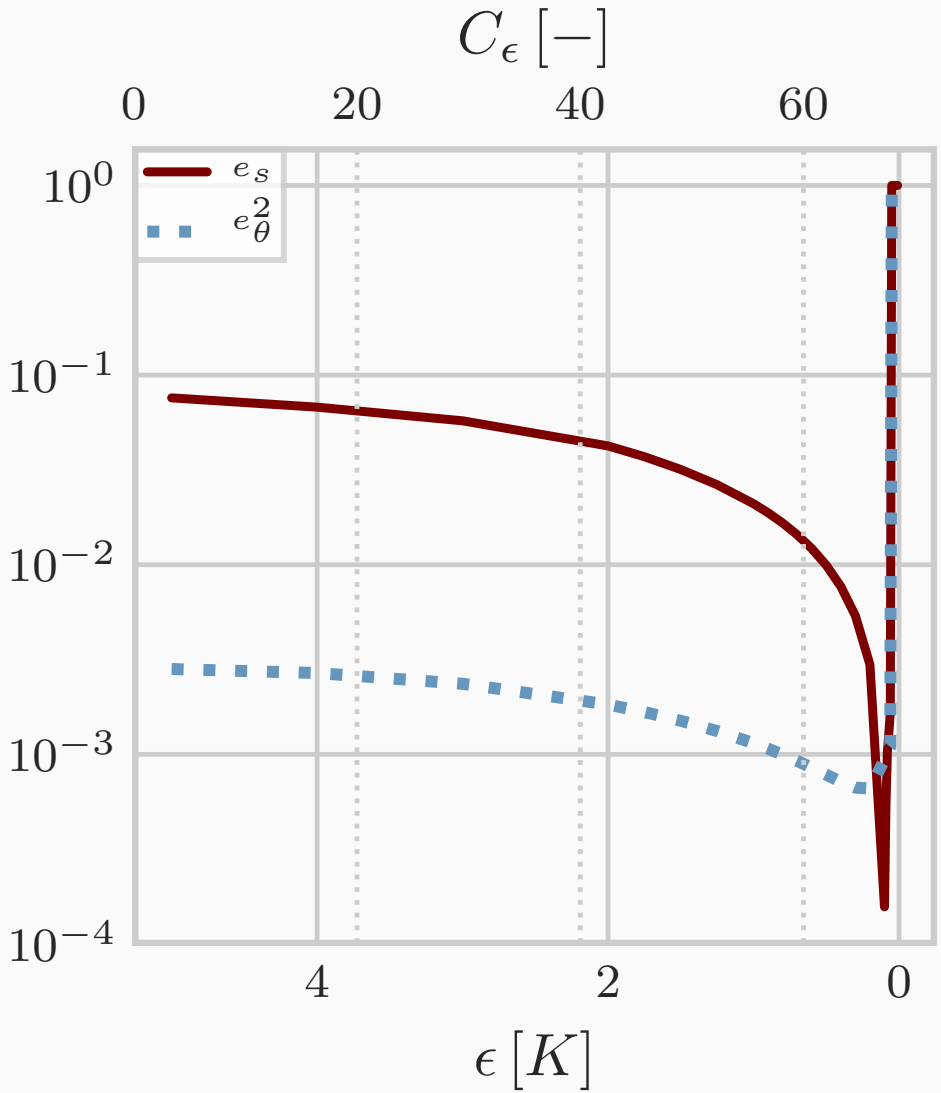
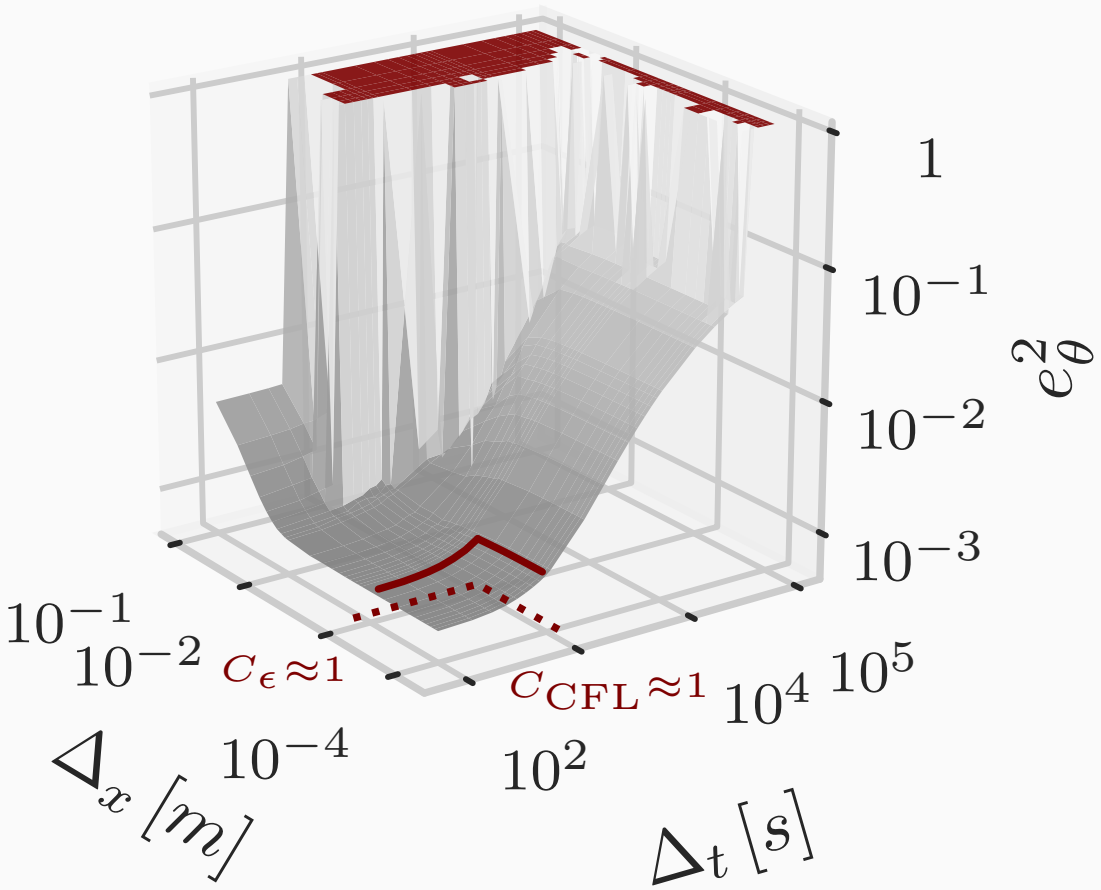


Figure: Velocity of the melting front

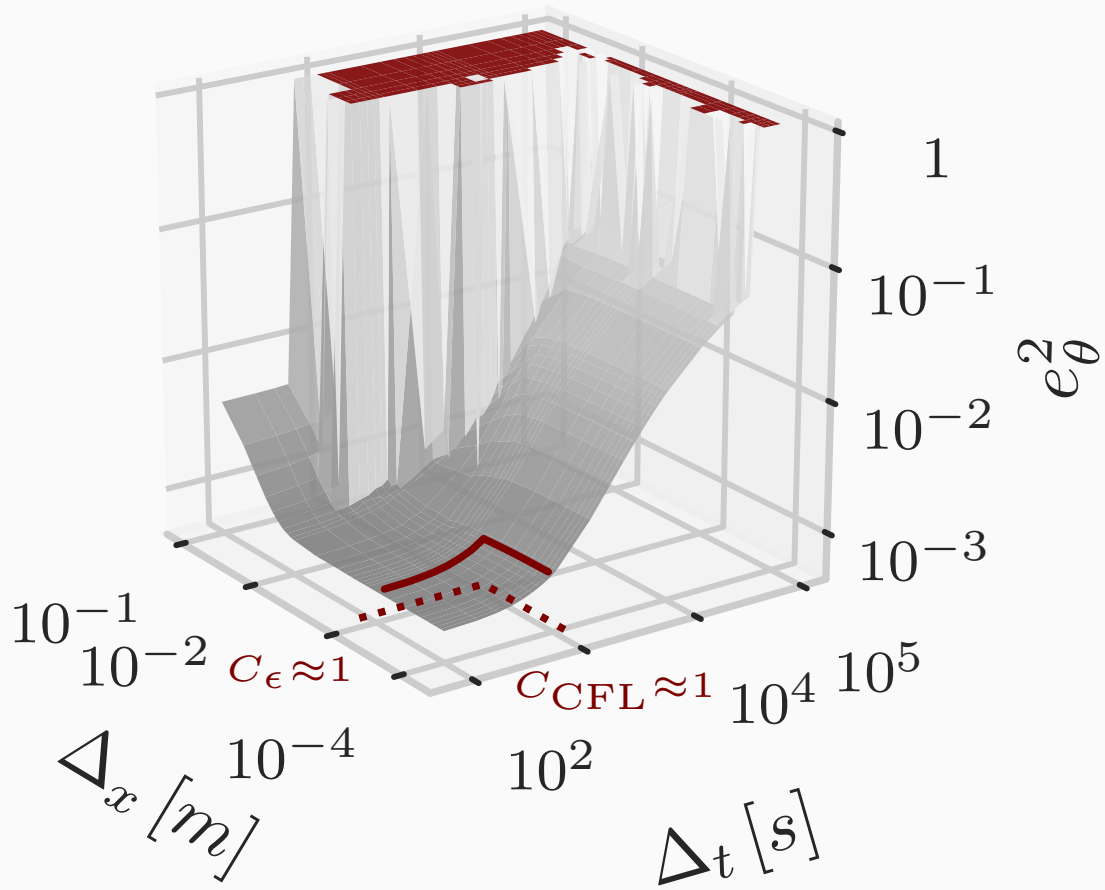
Disc. parameters	Method	e_θ^2	e_θ^∞	e_s
$\Delta_x{=}1.0{\cdot}10^{-2}\,\mathrm{m}$, $\epsilon{=}5.0{\cdot}10^{+0}\,\mathrm{K}$	EHC	$2.84{\cdot}10^{-3}$	$3.84{\cdot}10^{-3}$	$1.21{\cdot}10^{-2}$
$(\Delta_t{=}2.4{\cdot}10^{+3}\,\mathrm{s})$	TTM	$2.43{\cdot}10^{-3}$	$4.19{\cdot}10^{-3}$	$1.27{\cdot}10^{-2}$
$\Delta_x{=}1.0{\cdot}10^{-3}\,\mathrm{m}$, $\epsilon{=}5.0{\cdot}10^{-1}\,\mathrm{K}$	EHC	$7.81{\cdot}10^{-4}$	$8.28{\cdot}10^{-4}$	$1.59{\cdot}10^{-3}$
$(\Delta_t{=}2.4{\cdot}10^{+2}\,\mathrm{s})$	TTM	$7.41{\cdot}10^{-4}$	$8.50{\cdot}10^{-4}$	$1.24{\cdot}10^{-3}$
$\Delta_x{=}1.0{\cdot}10^{-4}\,\mathrm{m}$, $\epsilon{=}5.0{\cdot}10^{-2}\,\mathrm{K}$	EHC	$4.44{\cdot}10^{-4}$	$4.52{\cdot}10^{-4}$	$1.60{\cdot}10^{-5}$
$(\Delta_t{=}2.4{\cdot}10^{+1}\,\mathrm{s})$	TTM	$4.10{\cdot}10^{-4}$	$4.11{\cdot}10^{-4}$	$4.01{\cdot}10^{-5}$

Table: A sequence of the error norms for the $1d$ implementation of EHC and TTM methods at $t{=}t_{\max}$, for gradually increasing resolution.


$C_\epsilon\,[-]$
0
20
40
60
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
e_s
e_θ^2
4
2
0
$\epsilon\,[K]$

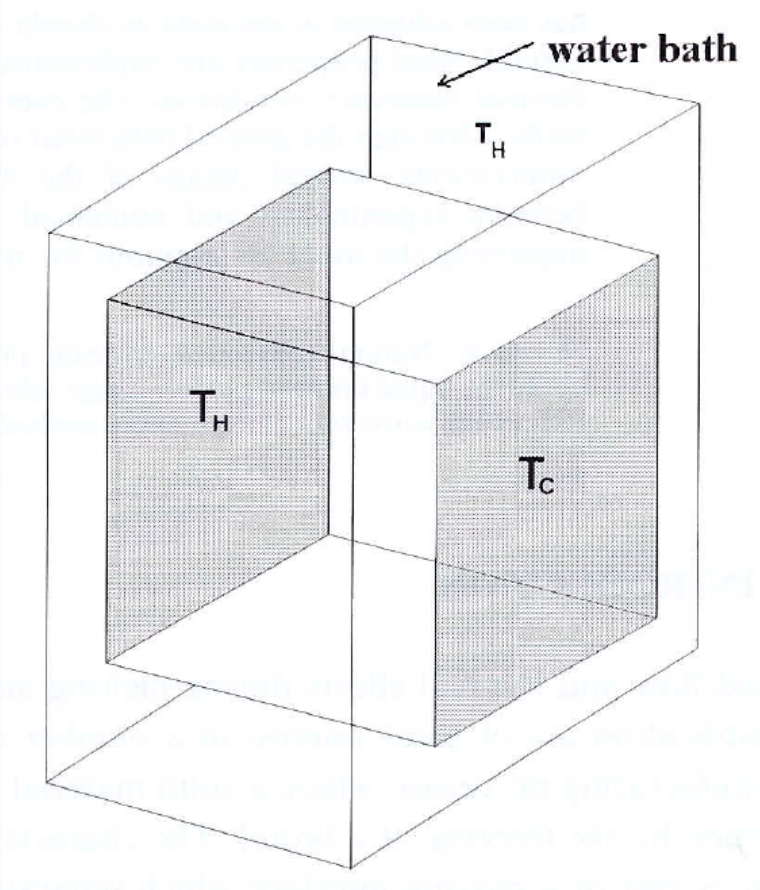

1
10^{-1}
10^{-2}
10^{-3}
e_θ^2
10^{-1}
10^{-2}
$C_\epsilon \approx 1$
10^{-4}
$\Delta_x\,[m]$
$C_{\mathrm{CFL}} \approx 1$
10^4
10^5
10^2
$\Delta_t\,[s]$

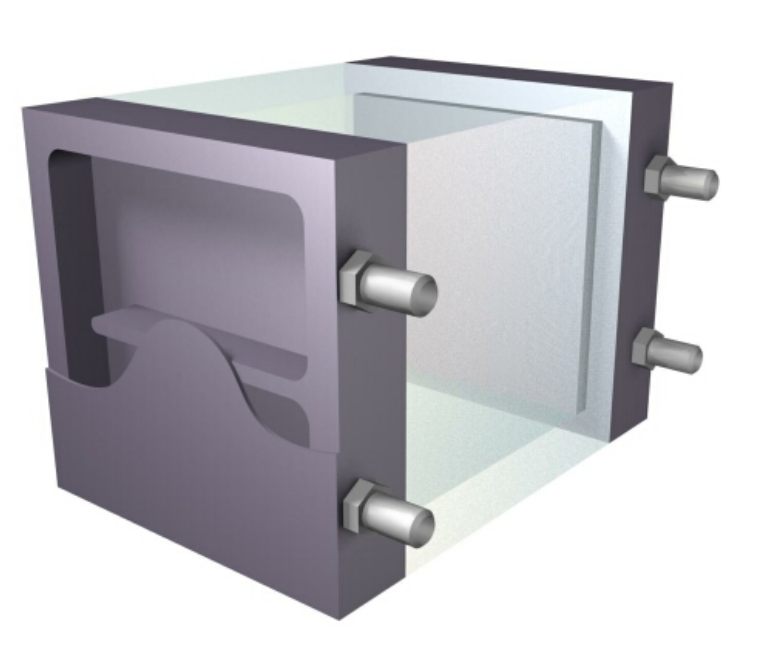
Benchmark code available at zenodo.org



Convective problem

Kowalewski benchmark[1]

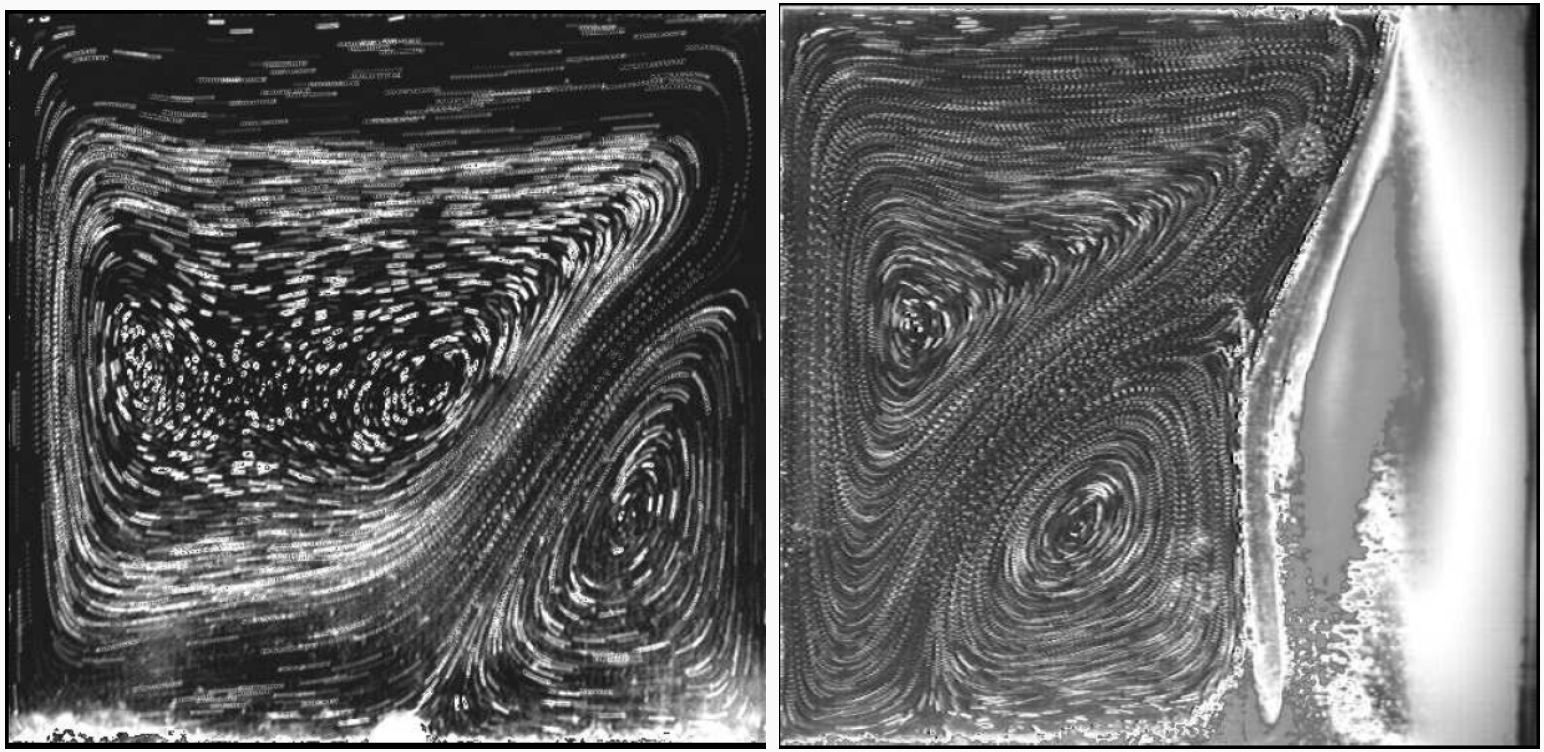




[1] Kowalewski, T. A. a Rebow, M. (1999). Freezing of Water in a Differentially Heated Cubic Cavity. International Journal of Computational Fluid Dynamics

Convective problem

Kowalewski benchmark[1]



(a) Natural convection

(b) Convection with phase change

[1] Kowalewski, T. A. a Rebow, M. (1999). Freezing of Water in a Differentially Heated Cubic Cavity. International Journal of Computational Fluid Dynamics

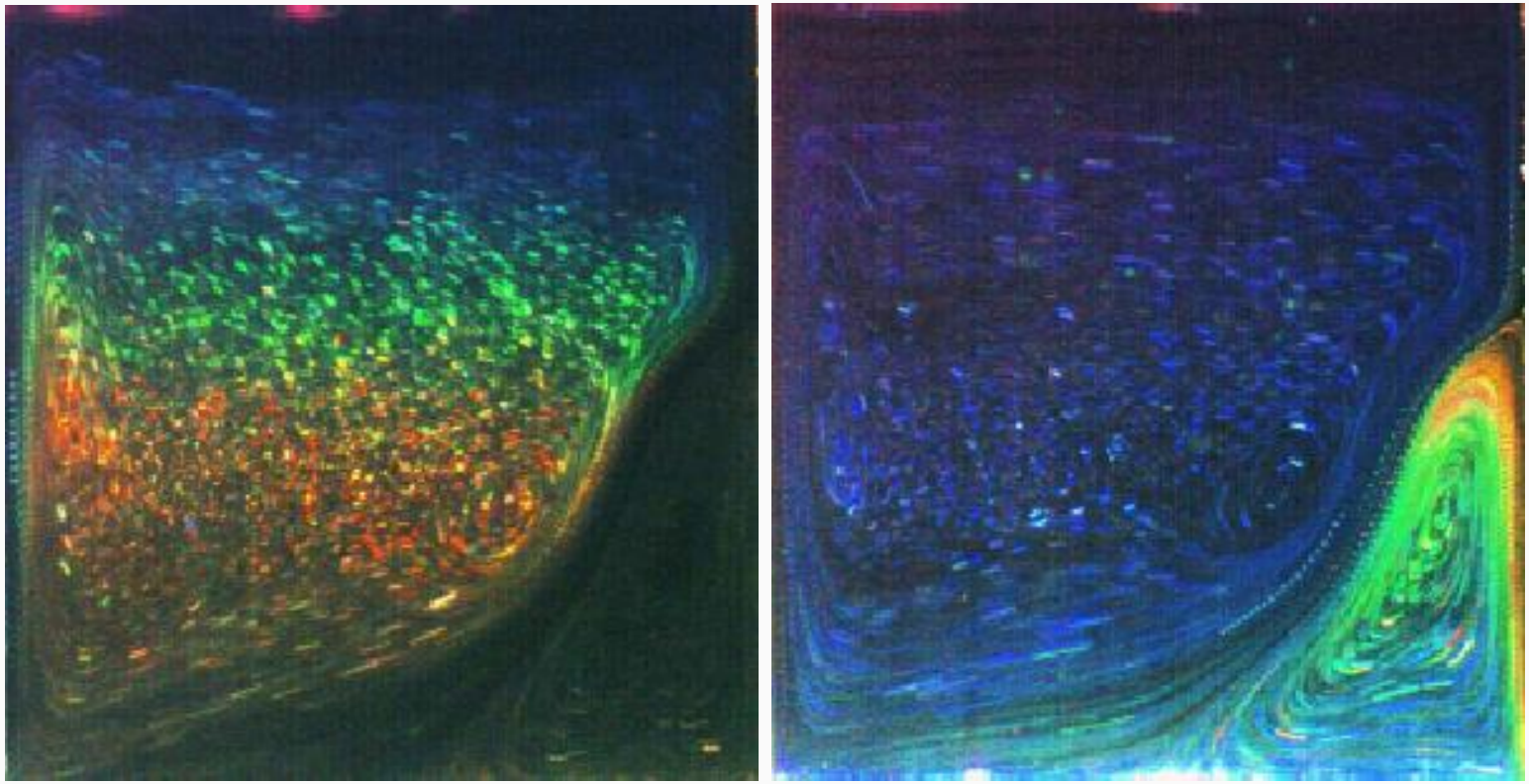


Figure: Temperature measurements

[1] Kowalewski, T. A. a Rebow, M. (1999). Freezing of Water in a Differentially Heated Cubic Cavity. International Journal of Computational Fluid Dynamics

Boussinesq approximation of thermal buyoancy:

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho\, \dot{\mathbf{v}} = -\nabla p + \operatorname{div}(2\mu \mathbb{D}) + \rho(\theta)\mathbf{g} \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

$$\rho\, c\, \dot{\theta} = \operatorname{div}(k \nabla \theta) \qquad \text{in } \omega^{\mathrm{l}} \cup \omega^{\mathrm{s}}$$

Boussinesq approximation with phase-change:

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega$$

$$\rho^{\text{eff}} \dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu^{\text{eff}} \mathbb{D}\right) + \rho^{\text{eff}}(\theta)\mathbf{g} \qquad \text{in } \omega$$

$$\rho^{\text{eff}} c^{\text{eff}} \dot{\theta} = \operatorname{div}\left(k^{\text{eff}} \nabla \theta\right) \qquad \text{in } \omega$$

Boussinesq approximation with phase-change (mollified params):

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)} \dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu_\epsilon^{(1)} \mathbb{D}\right) + \rho_\epsilon^{(1)}(\theta)\mathbf{g} \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)} c_\epsilon^{(1)} \dot{\theta} = \operatorname{div}\left(k_\epsilon^{(1)} \nabla \theta\right) \qquad \text{in } \omega$$

Boussinesq approximation with phase-change (mollified params):

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)} \dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu_\epsilon^{(1)} \mathbb{D}\right) + \rho_\epsilon^{(1)}(\theta)\mathbf{g} \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)} c_\epsilon^{(1)} \dot{\theta} = \operatorname{div}\left(k_\epsilon^{(1)} \nabla \theta\right) \qquad \text{in } \omega$$

Velocity correction in ice-phase (viscosity contrast):

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)} \dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu_\epsilon^{(1)} \mathbb{D}\right) + \rho_\epsilon^{(1)}(\theta)\mathbf{g} + \tau \chi_\epsilon^{(1),\mathrm{s}}(\theta)\mathbf{v} \qquad \text{in } \omega$$

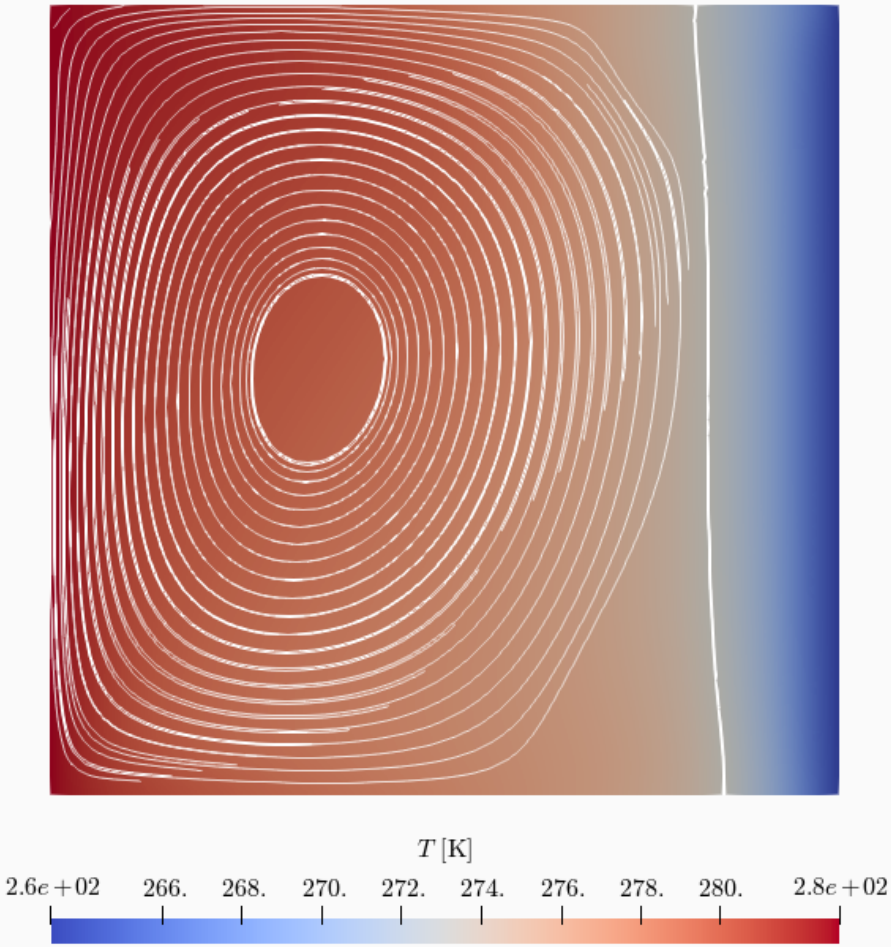
$$\rho_\epsilon^{(1)} c_\epsilon^{(1)} \dot{\theta} = \operatorname{div}\left(k_\epsilon^{(1)} \nabla \theta\right) \qquad \text{in } \omega$$

Velocity correction in ice-phase (Carman-Kozeny penalty):

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \omega$$

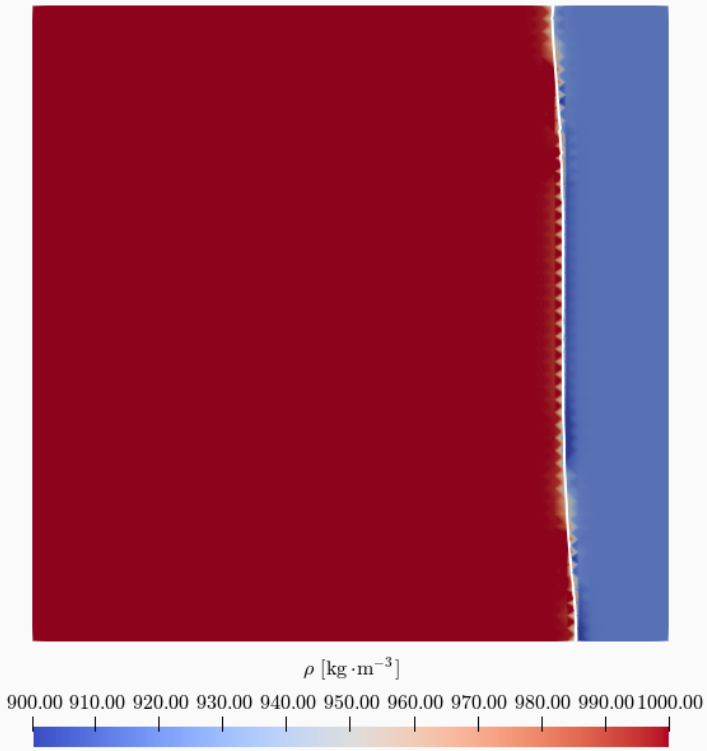
$$\rho_\epsilon^{(1)} \dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu_\epsilon^{(1)} \mathbb{D}\right) + \rho_\epsilon^{(1)}(\theta)\mathbf{g} + \tau \chi_\epsilon^{(1),\mathrm{s}}(\theta)\mathbf{v} \qquad \text{in } \omega$$

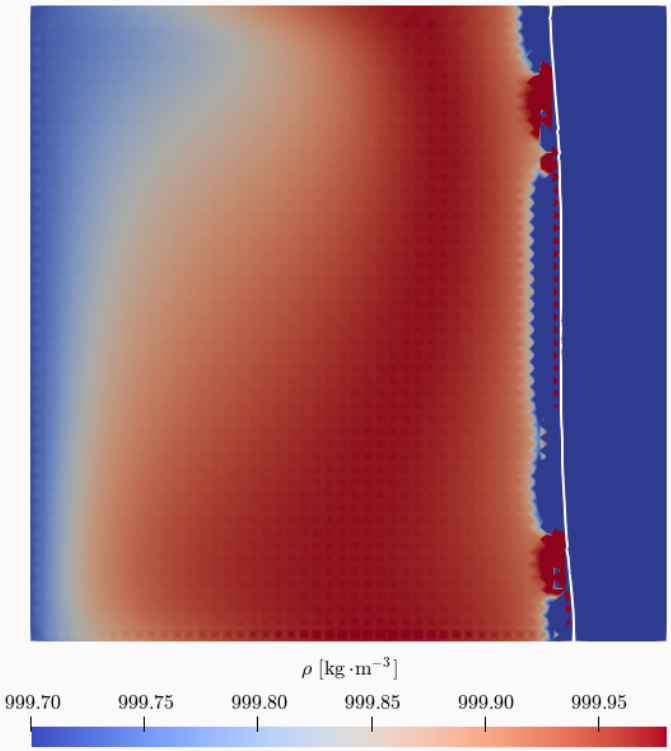
$$\rho_\epsilon^{(1)} c_\epsilon^{(1)} \dot{\theta} = \operatorname{div}\left(k_\epsilon^{(1)} \nabla \theta\right) \qquad \text{in } \omega$$


T [K]
2.6e+02
266.
268.
270.
272.
274.
276.
278.
280.
2.8e+02

Convective problem

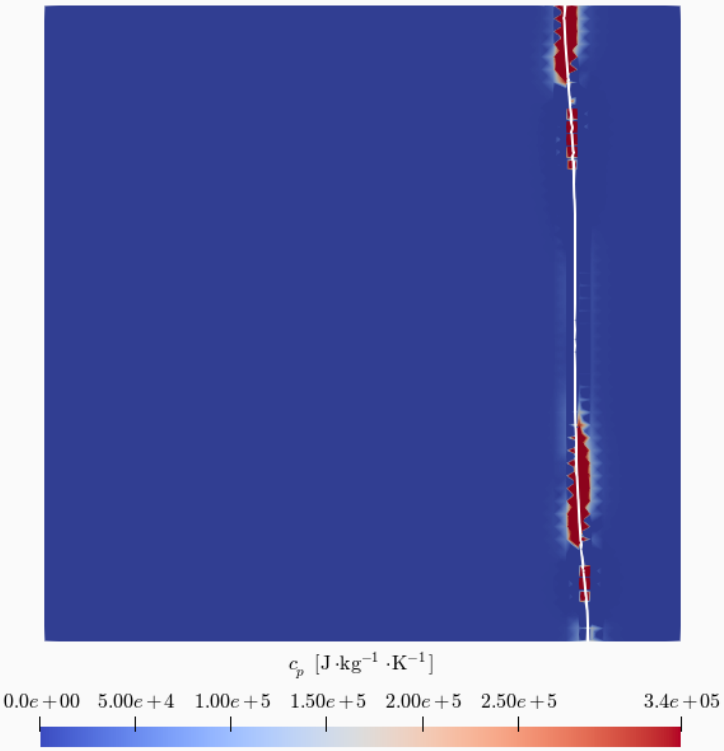
Results

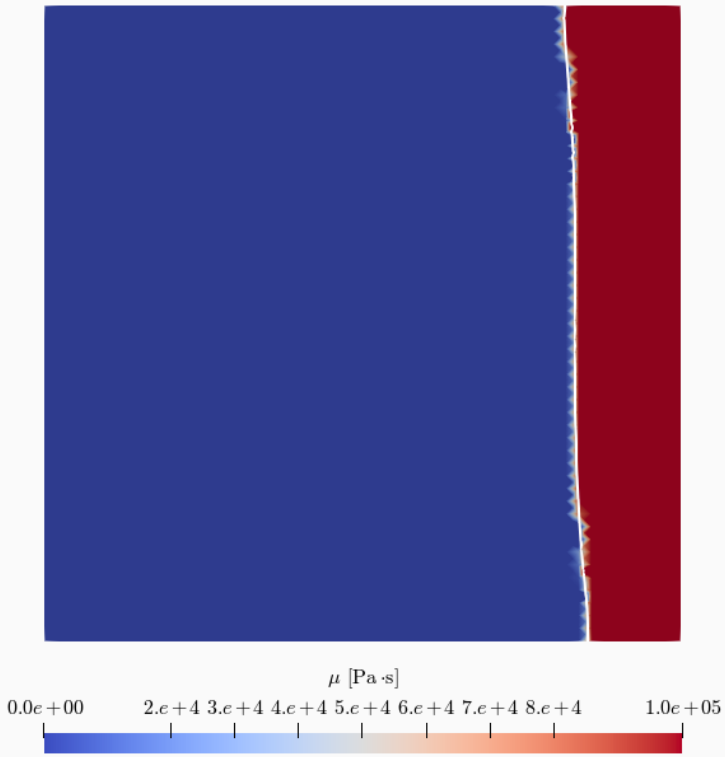


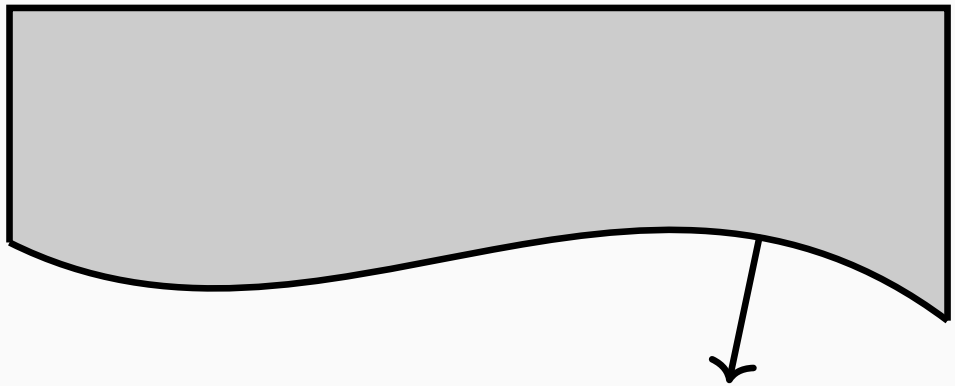


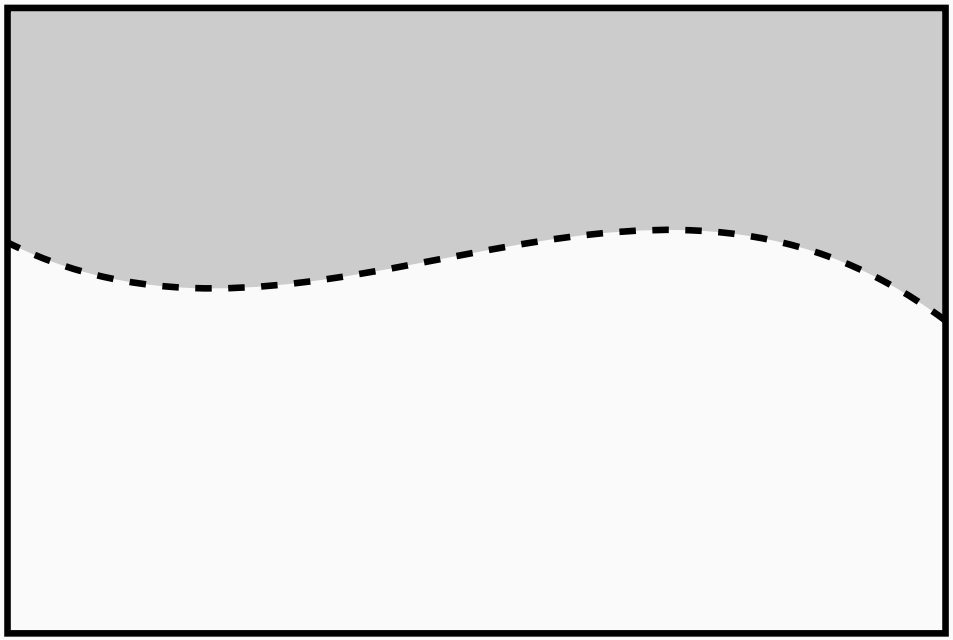
Convective problem

Results









One-phase problem governing equations:

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \omega^{\mathrm{s}}$$

$$\rho^{\mathrm{s}}\dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu^{\mathrm{s}}\mathbb{D}\right) + \rho^{\mathrm{s}}\mathbf{g} \qquad \text{in } \omega^{\mathrm{s}}$$

$$\rho^{\mathrm{s}}c^{\mathrm{s}}\dot{\theta} = \operatorname{div}\left(k^{\mathrm{s}}\nabla\theta\right) \qquad \text{in } \omega^{\mathrm{s}}$$

Boundary conditions:

$$\mathbf{v} = \mathbf{0}\,, \qquad \theta = \theta_{\mathrm{top}} \qquad \text{on } \gamma_{\mathrm{top}}$$

$$\mathbf{v}\cdot\mathbf{n} = 0\,, \mathbb{T}\mathbf{n}\big|_{\tau} = \mathbf{0}\,, \quad \nabla\theta\cdot\mathbf{n} = 0 \qquad \text{on } \gamma_{\mathrm{left}} \cup \gamma_{\mathrm{right}}$$

$$\mathbb{T}\mathbf{n} = -p_{\mathrm{bot}}(z)\mathbf{n}\,, \qquad \theta = \theta_{\mathrm{m}} \qquad \text{on } \gamma_{\mathrm{bot}}$$

Kinematic condition for the bottom boundary:

$$\frac{\partial h}{\partial t} = \sqrt{h_x^2 + 1}\left(\left(\frac{\mathbf{q}_{\mathrm{in}} + k^{\mathrm{s}}\nabla\theta}{\rho^{\mathrm{s}}\mathrm{L}_{\mathrm{m}}} - \mathbf{v}\right)\cdot\mathbf{n}\right) \qquad \text{on } \gamma_{\mathrm{bot}}$$

Two-phase problem:

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)}\dot{\mathbf{v}} = -\nabla p + \operatorname{div}\left(2\mu_\epsilon^{(1)}\mathbb{D}\right) + \rho_\epsilon^{(1)}\mathbf{g} + \tau\chi_\epsilon^{(1),\mathrm{l}}(\theta)\mathbf{v} \qquad \text{in } \omega$$

$$\rho_\epsilon^{(1)}c_\epsilon^{(1)}\dot{\theta} = \operatorname{div}\left(\mathbb{K}_\epsilon^{(1)}\nabla\theta\right) \qquad \text{in } \omega$$

$$\mathbb{K}_\epsilon^{(1)}(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & K^{\mathrm{l}} \end{pmatrix}\chi_\epsilon^{(1),\mathrm{l}}(\theta) + \begin{pmatrix} k^{\mathrm{s}} & 0 \\ 0 & k^{\mathrm{s}} \end{pmatrix}\chi_\epsilon^{(1),\mathrm{s}}(\theta)$$

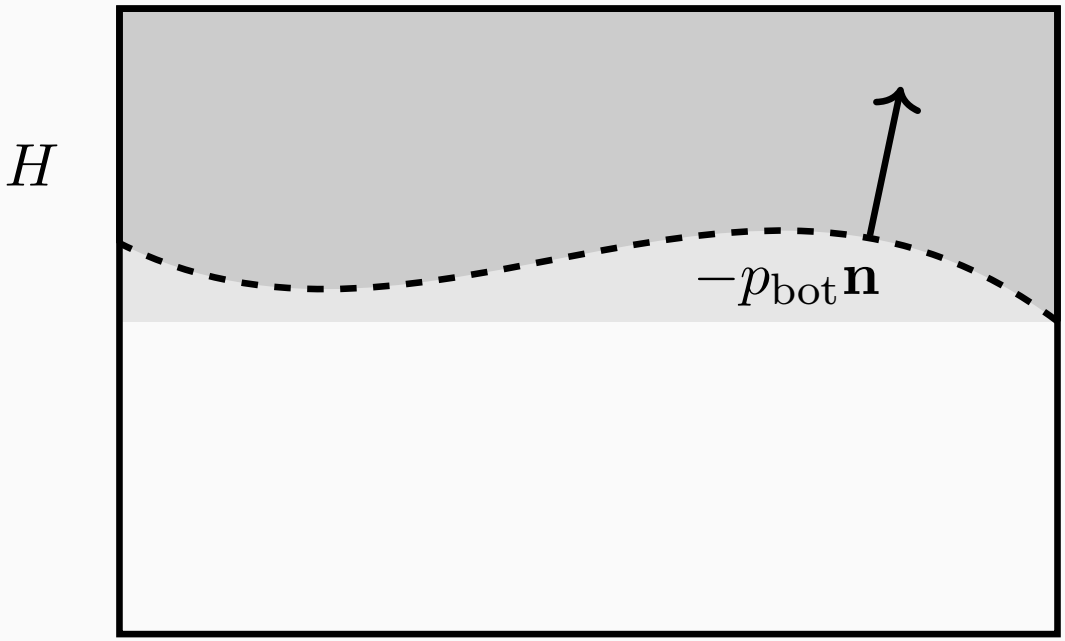
Boundary conditions:

$$\mathbf{v} = \mathbf{0}\,, \qquad \theta = \theta_{\mathrm{top}} \qquad \text{on } \gamma_{\mathrm{top}}$$

$$\mathbf{v}\cdot\mathbf{n} = 0\,, \mathbb{T}\mathbf{n}\big|_\tau = \mathbf{0}\,, \quad \nabla\theta\cdot\mathbf{n} = 0 \qquad \text{on } \gamma_{\mathrm{left}} \cup \gamma_{\mathrm{right}}$$

$$\mathbf{v}\cdot\mathbf{n} = 0\,, \mathbb{T}\mathbf{n}\big|_\tau = \mathbf{0}\,, \quad k^{\mathrm{s}}\nabla\theta = \mathbf{q}_{\mathrm{in}} \qquad \text{on } \gamma_{\mathrm{bot}}$$


H



$$p_{\mathrm{bot}}(z) = (\rho^{\mathrm{s}} H - \rho^{\mathrm{l}}(z - H))g$$

Implementation of one-phase simulation:

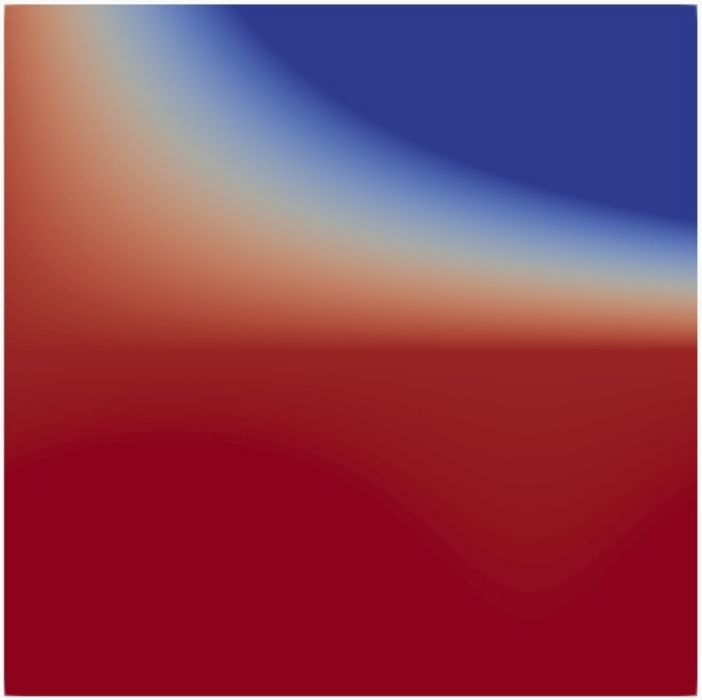
- kinematic description - ALE method

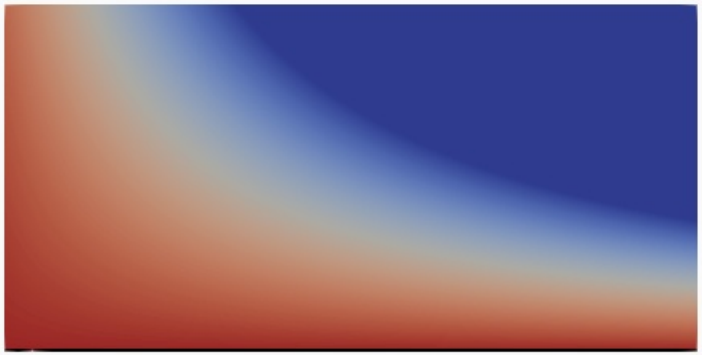
Implementation of one-phase simulation:

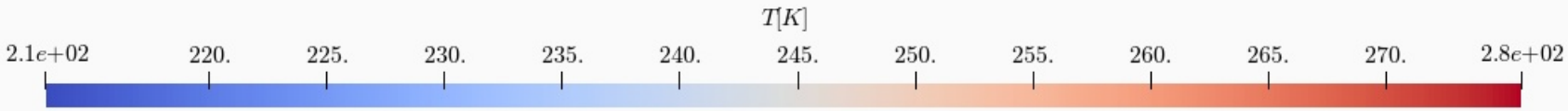
- kinematic description - ALE method
- bottom boundary kinematic equation - the Nitsche method

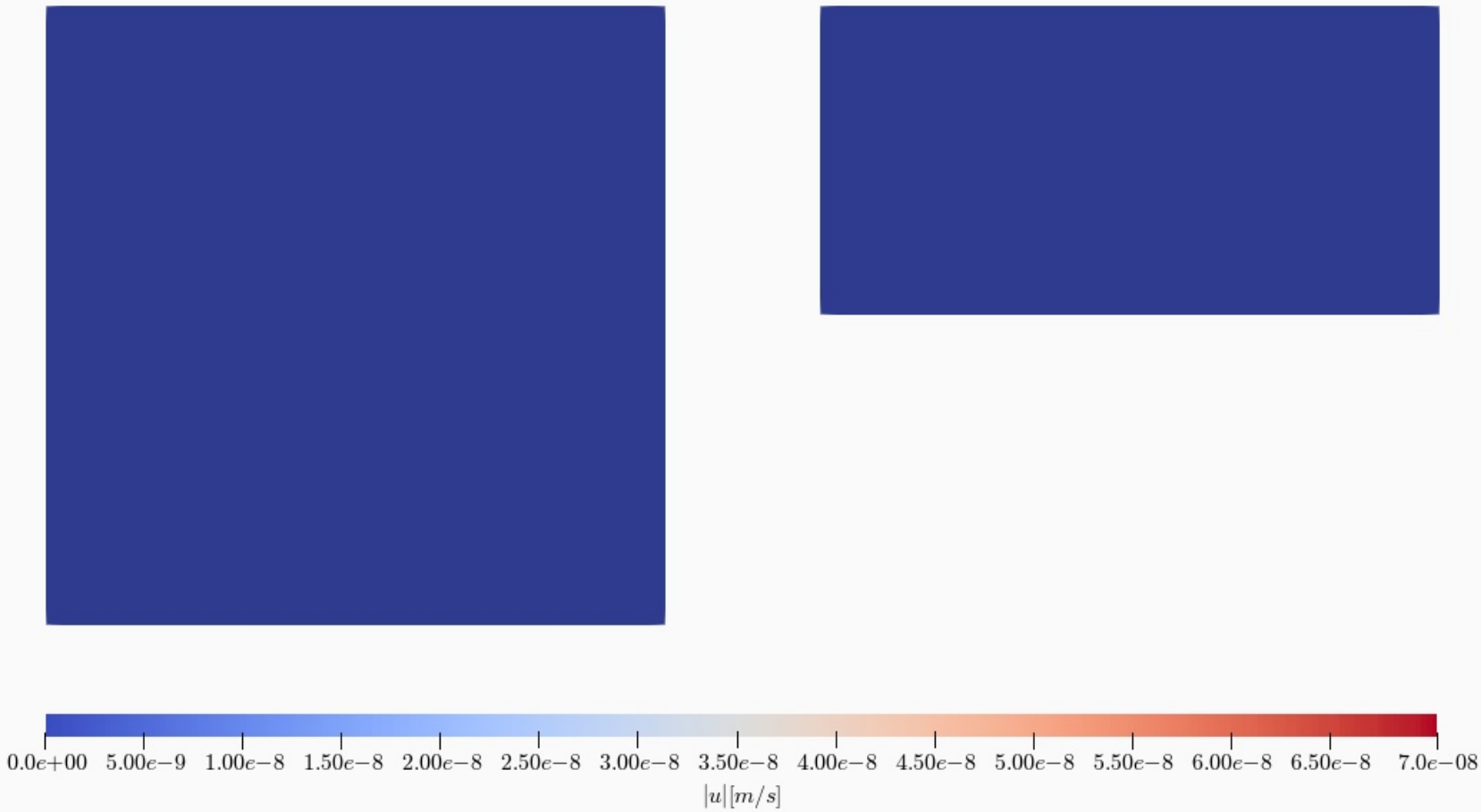
Implementation of one-phase simulation:

- kinematic description - ALE method
- bottom boundary kinematic equation - the Nitsche method
- velocity correction for water






T[K]
2.1e+02
220.
225.
230.
235.
240.
245.
250.
255.
260.
265.
270.
2.8e+02


0.0e+00 5.00e−9 1.00e−8 1.50e−8 2.00e−8 2.50e−8 3.00e−8 3.50e−8 4.00e−8 4.50e−8 5.00e−8 5.50e−8 6.00e−8 6.50e−8 7.0e−08
|u|[m/s]

Outlook

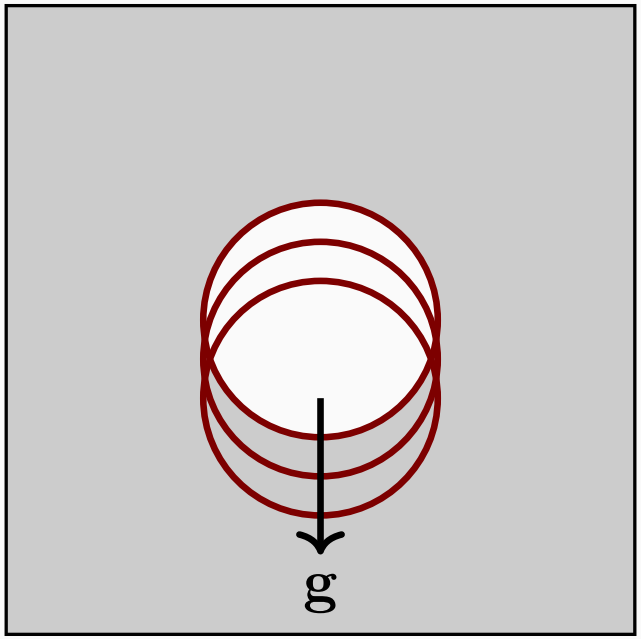
Next steps:

- one-phase ice shell evolution - (real) temperature dependent viscosities for ice

Outlook

Next steps:

- one-phase ice shell evolution - (real) temperature dependent viscosities for ice
- two-phase simulation - melting probe problem



Thank you for your attention!

