EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research,
Development and Education

MINISTRY OF EDUCATION,
YOUTH AND SPORTS

Numerical model of a cooling magma chamber

Vojtěch Patočka

Nicola Tosi, Enrico Calzavarini, and Václav Špillar

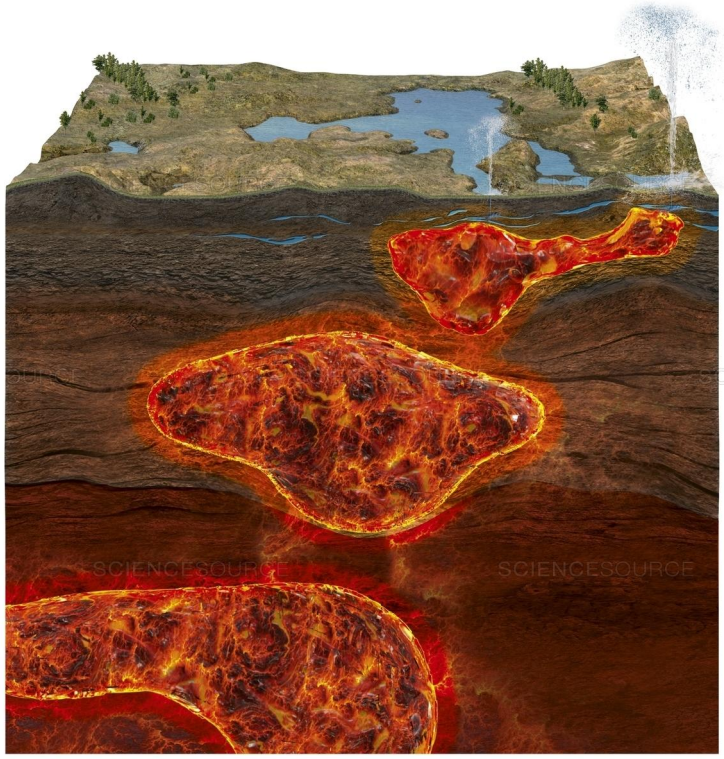


Content

- Parametrized model J&W, 1994
  a) Gov. Eqs
  b) Assumptions
  c) Results

- Caveats of a direct model
  a) Convective vigor of magma chambers
  b) Latent heat – key player?
  c) Wall rock behavior
  d) Sediment deposition
  e) Preliminary results

does it look like this?



credits: SCIENCESOURCE

PHYSICAL REVIEW FLUIDS **5**, 114304 (2020)

Editors' Suggestion

Settling of inertial particles in turbulent Rayleigh-Bénard convection

Vojtěch Patočka,[1,*] Enrico Calzavarini,[2] and Nicola Tosi[1]



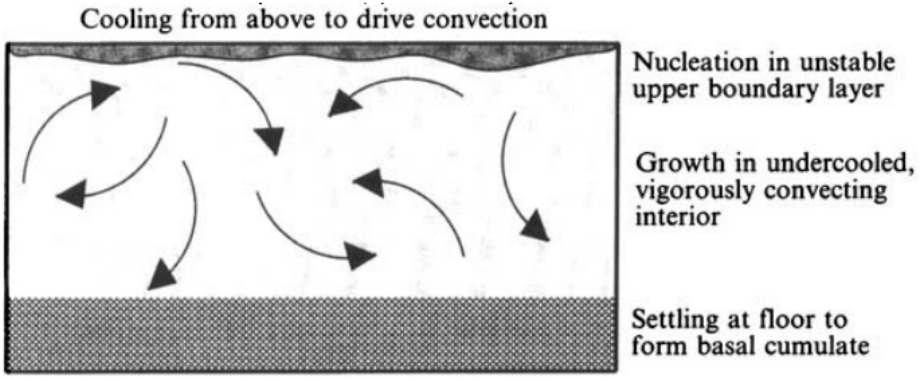
J. Fluid Mech. (1994), *vol.* 273, *pp.* 83–107
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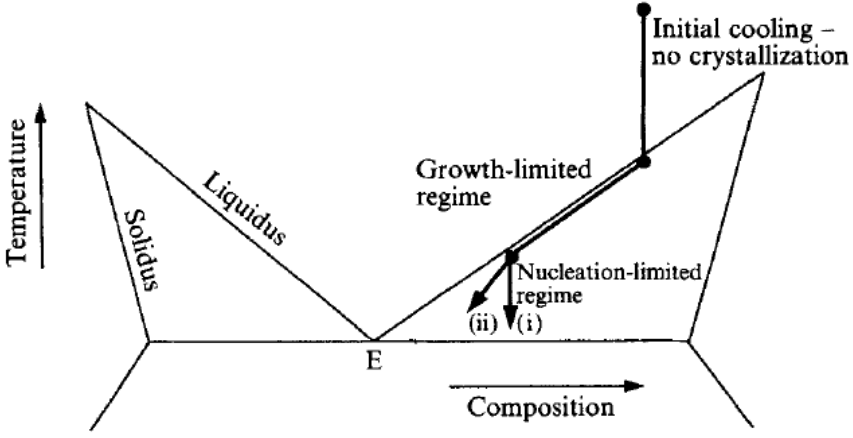
The nucleation, growth and settling of crystals from a turbulently convecting fluid

By RICHARD A. JARVIS[1]† AND ANDREW W. WOODS[2]

[1] Department of Geology and Geophysics, Yale University, New Haven, CT 06511, USA

[2] Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

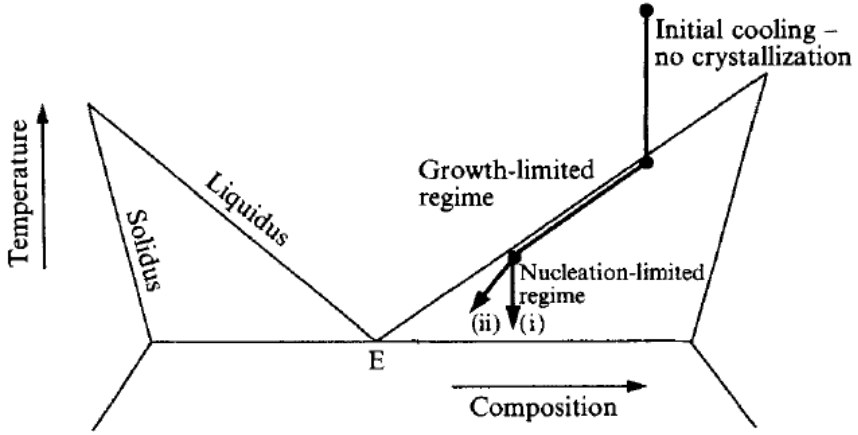




Parametrized model

$$(H-h)\,\rho c_p \frac{\mathrm{d}T}{\mathrm{d}t} = -\mathscr{F} + \rho \mathscr{L}(H-h)\,R_P,$$

Conservation of mass in the bulk convecting fluid takes the form



$$(1-\Phi)\,\mathrm{d}C/\mathrm{d}t = (C-C_s)\,R_P,$$

$$C_s - C_\alpha = k_D(C-C_\alpha);$$

$$T_L = T_\alpha + m(C-C_\alpha),$$

$$\mathrm{d}T_L/\mathrm{d}t = -(1-k_D)(T_\alpha - T_L)\,R_P,$$

$$V(t) \equiv V(T_L - T).$$

Bulk

$$N(t) \equiv \frac{Bh_b(t)}{H - h(t)} \tilde{N}(T_L - T_R),$$

Roof

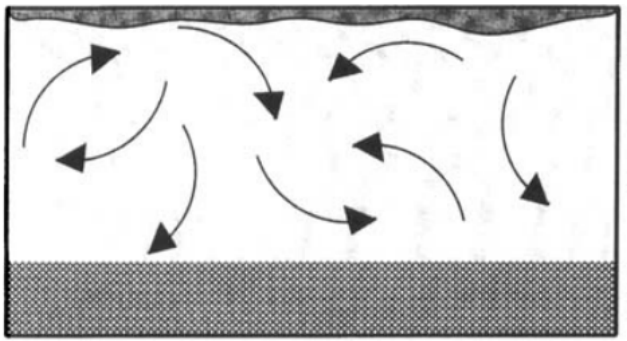
Settling model: Martin & Nokes, 1989

Suspension: equilibrium size distribution

$$\phi(a,t) = \frac{N(t)}{V(t)} \exp\left[-\frac{\lambda(t)\, a^3}{3(H - h(t))\, V(t)} \right],$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial (V\phi)}{\partial a} = -\frac{\lambda a^2}{H - h}\phi,$$

$$(H - h)\, R_P = R_S = 4\pi (H - h)^2\, NV/\lambda,$$

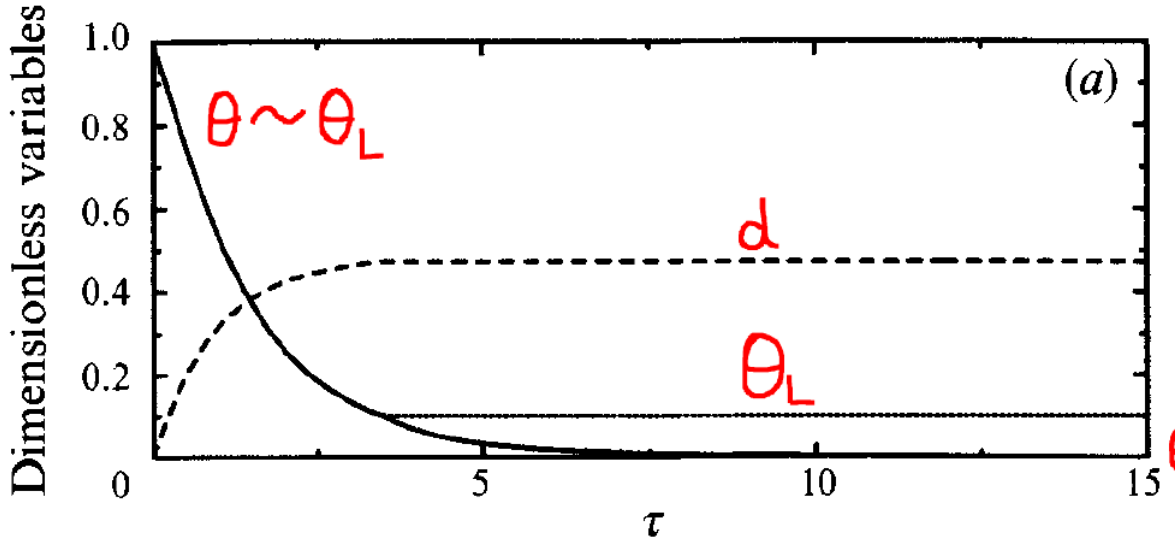


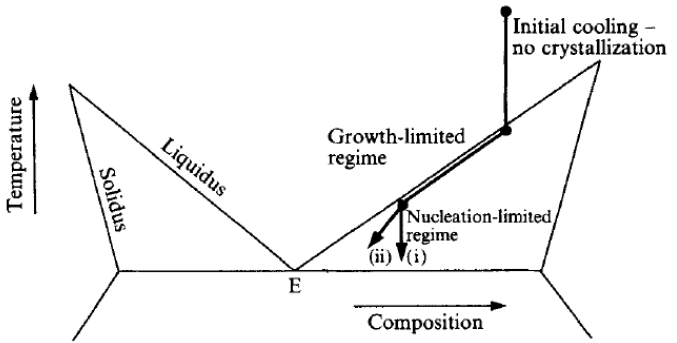
Growth and sedimentation rates

$$R_P(t) = 4\pi V(t) \int_0^{a(0;t)} a^2 \phi(a,t)\, \mathrm{d}a;$$

$$R_S(t) = \tfrac{4}{3}\pi \lambda(t) \int_0^{a(0;t)} a^5 \phi(a,t)\, \mathrm{d}a,$$

Constant roof temperature



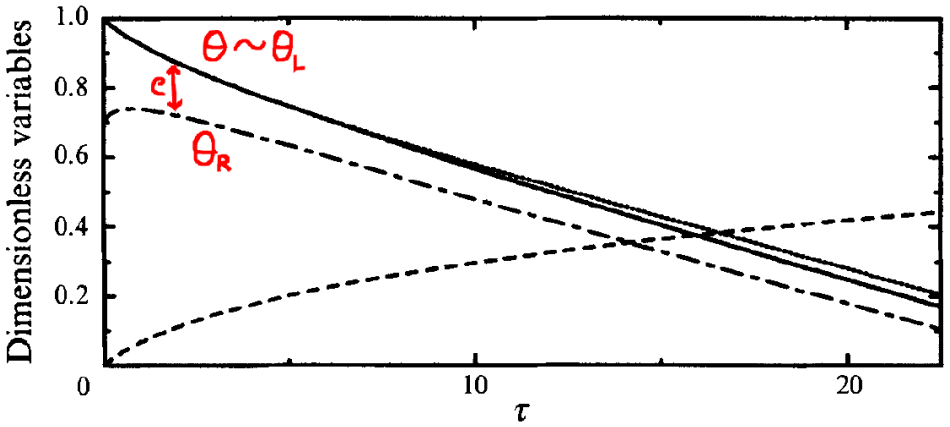


Evolution of non-dimensional quantities:

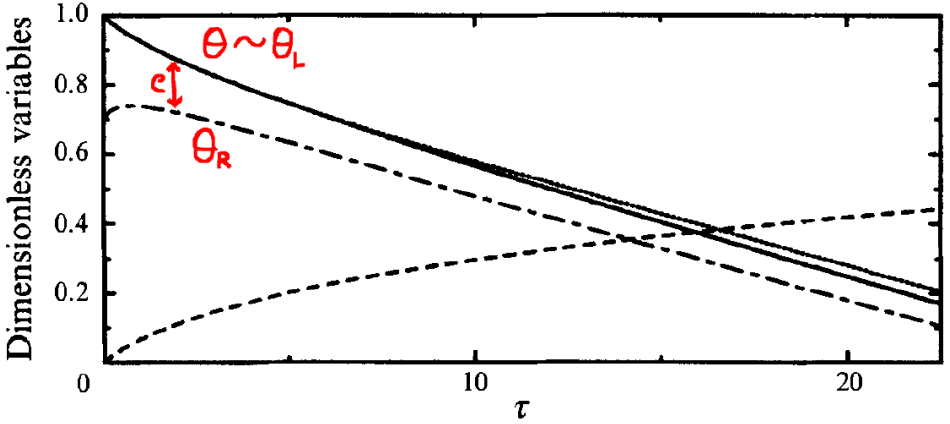
1) Growth limited regime: bulk temperature close to liquidus, roof under-cooled

2) Nucleation limited regime: (liquidus - roof temperature) ~ nucleation delay

Evolving roof temperature



Evolving roof temperature



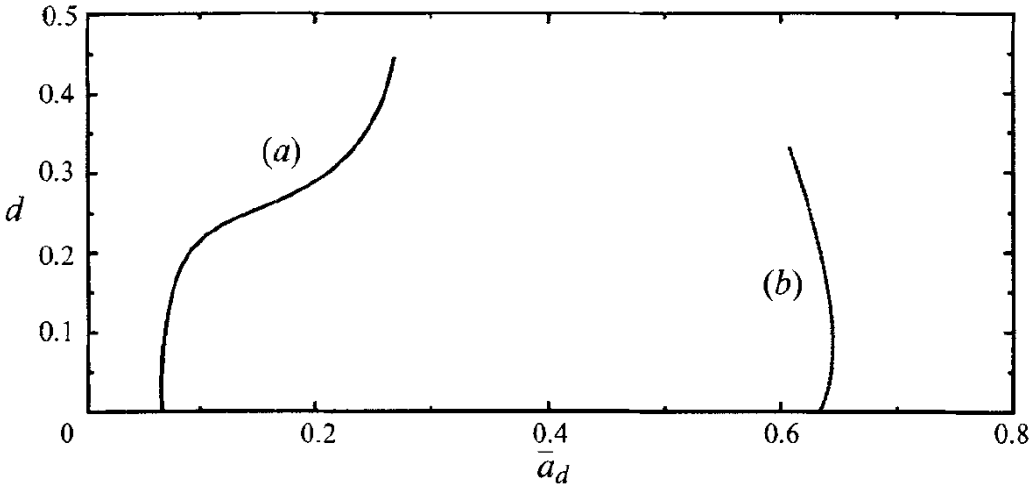
Particle size distribution in sediment:

1) normal grading (small over large)

2) inverse grading

Assumption: Settling ~ Stokes velocity

$$\bar{a}_d(t) = \int_0^{a(0;t)} a^3 \phi(a,t)\,\mathrm{d}a \Big/ \int_0^{a(0;t)} a^2 \phi(a,t)\,\mathrm{d}a$$



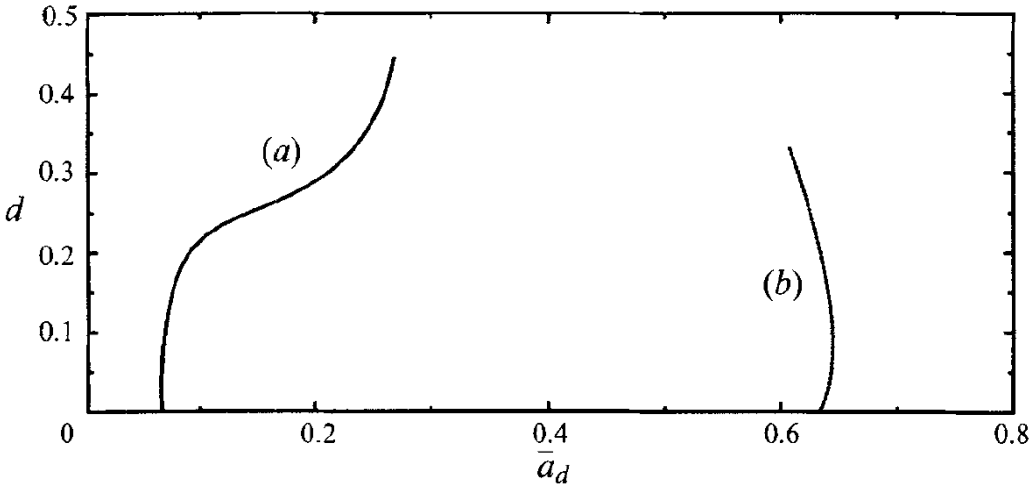
Size grading



Particle size distribution in sediment:

1) normal grading (small over large)

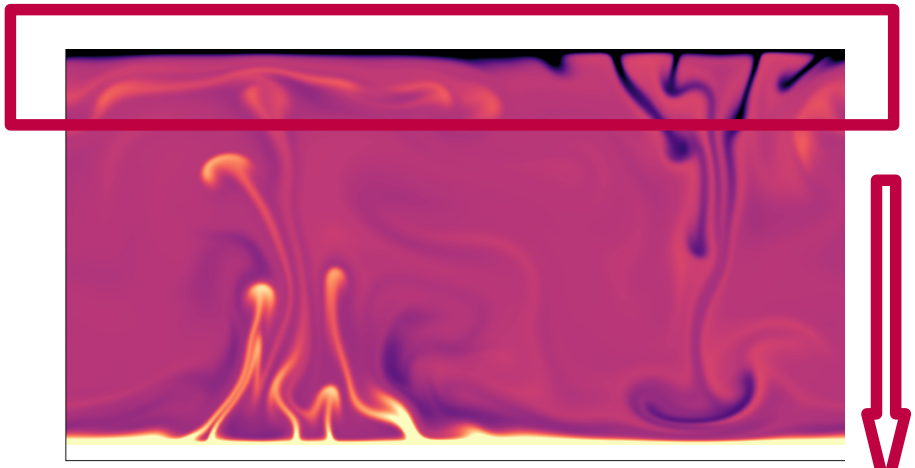
2) inverse grading



J&W1994 weaknesses

- Idealized view of the thermal boundary layer
- Bulk fluid is a statistical pool with no depth
- Simple settling model of Martin & Nokes for all crystals
- Simplified boundary conditions
- Linear crystal growth kinetics

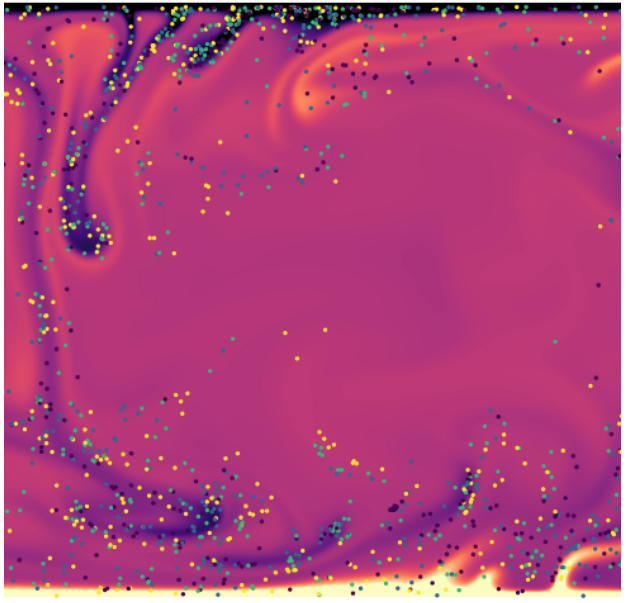
Direct model: nucleation and growth



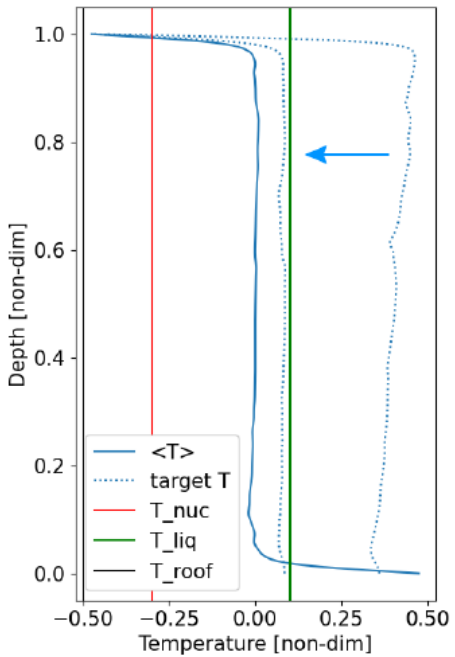
$$\tilde{N}(\Delta T) = \tilde{N}_{\mathrm{ref}} \left(\frac{T_{\mathrm{L}}(h,\Phi) - T - E}{\Delta T_{\mathrm{ref}} - E} \right)^{p}$$

$$V(\Delta T) = \frac{\mathrm{d}r_{\mathrm{p}}}{\mathrm{d}t} = V_{\mathrm{ref}} \left(\frac{T_{\mathrm{L}}(h,\Phi) - T_{\mathrm{p}}}{\Delta T_{\mathrm{ref}}} \right)^{q}$$

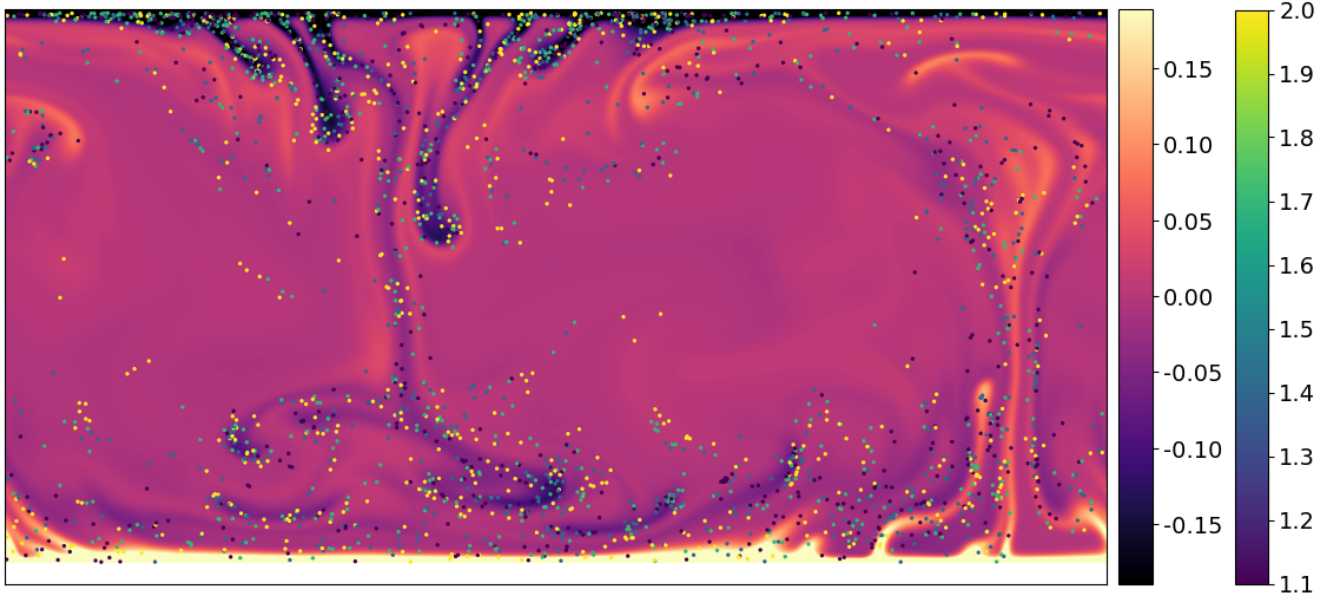




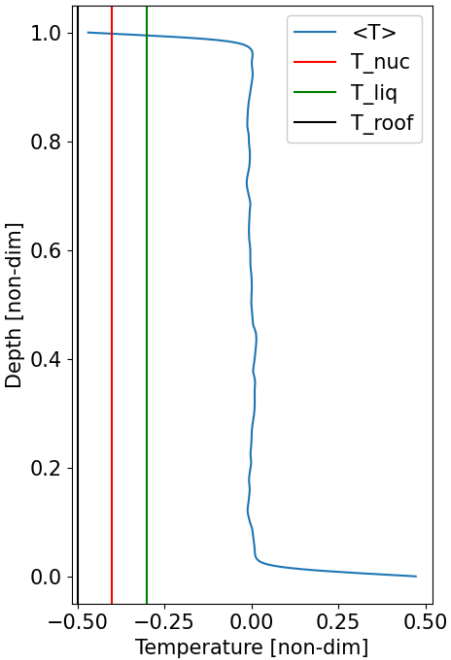
Particles are nucleated only for temperatures below T_{nuc} and grow in size if $T < T_{\mathrm{liq}}$. In Res: TBL only the TBL is below the liquidus (first stage of cooling), while in Res: bulk the situation is depicted by the solid lines below (second stage, $\bar{T} < T_{\mathrm{liq}}$).

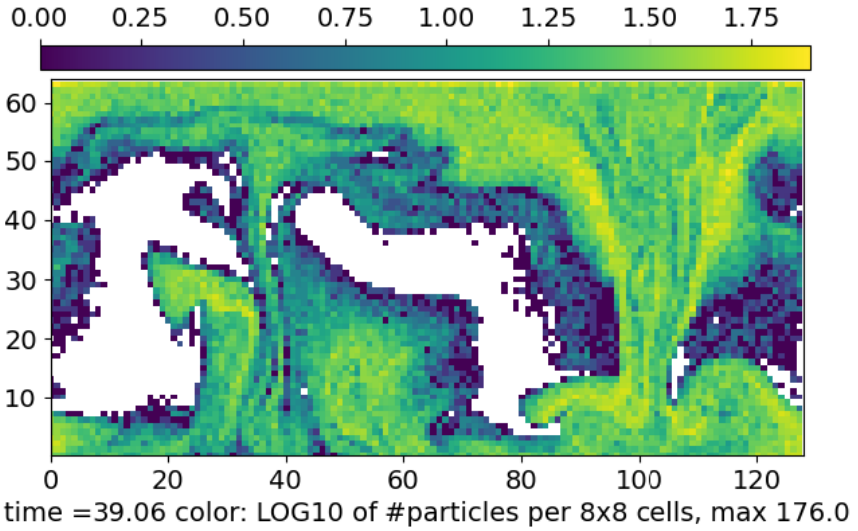


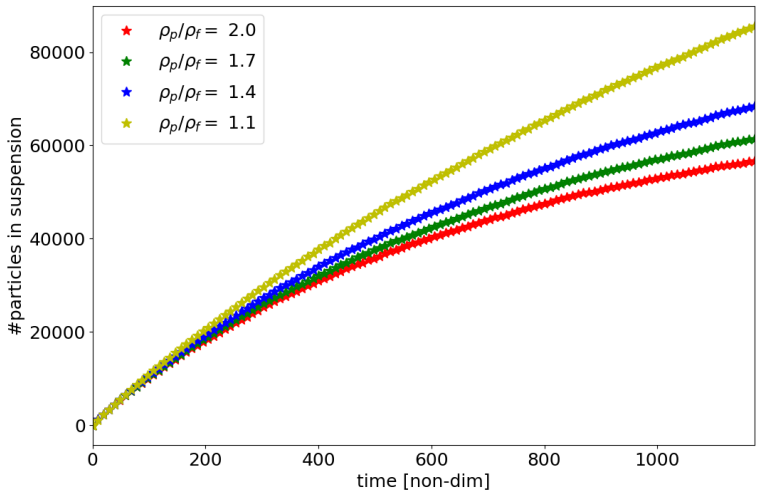
Temperature field and suspended particles. Time t=58.6, $Ra = 10^8$, $Pr = 50$.

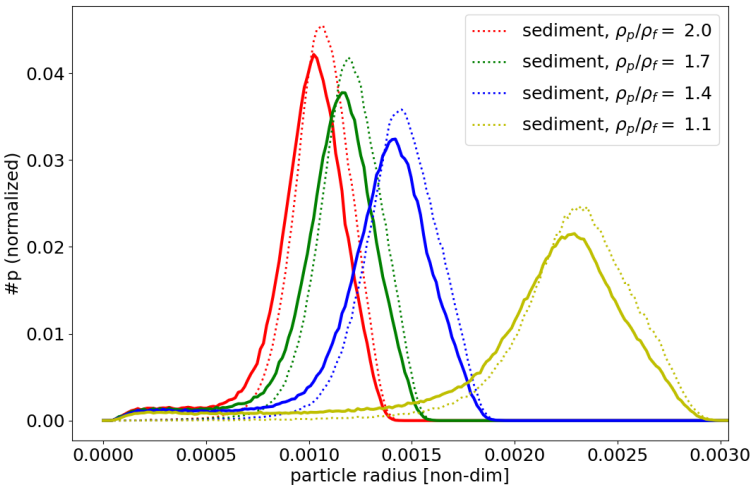


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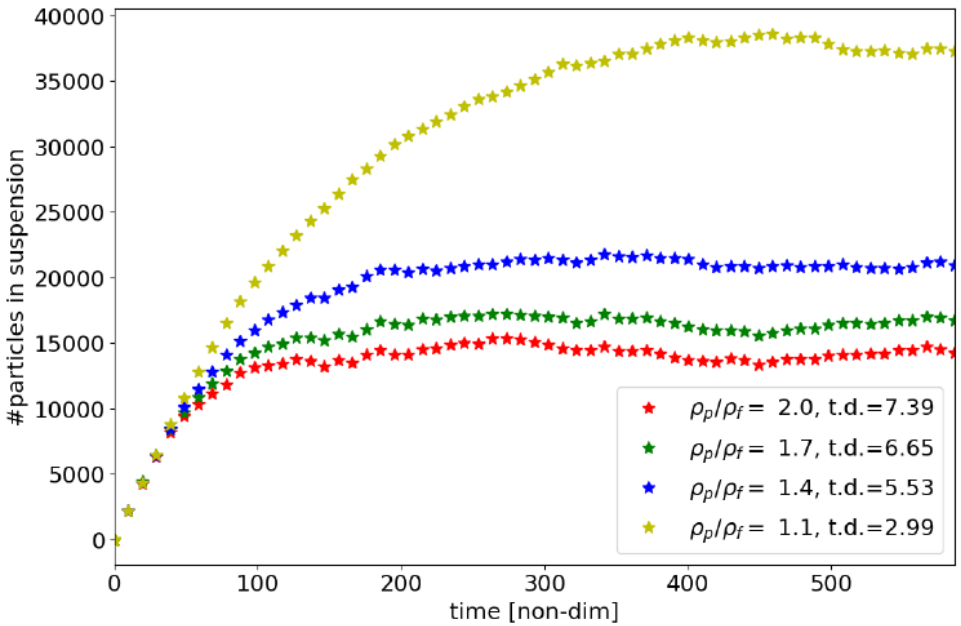


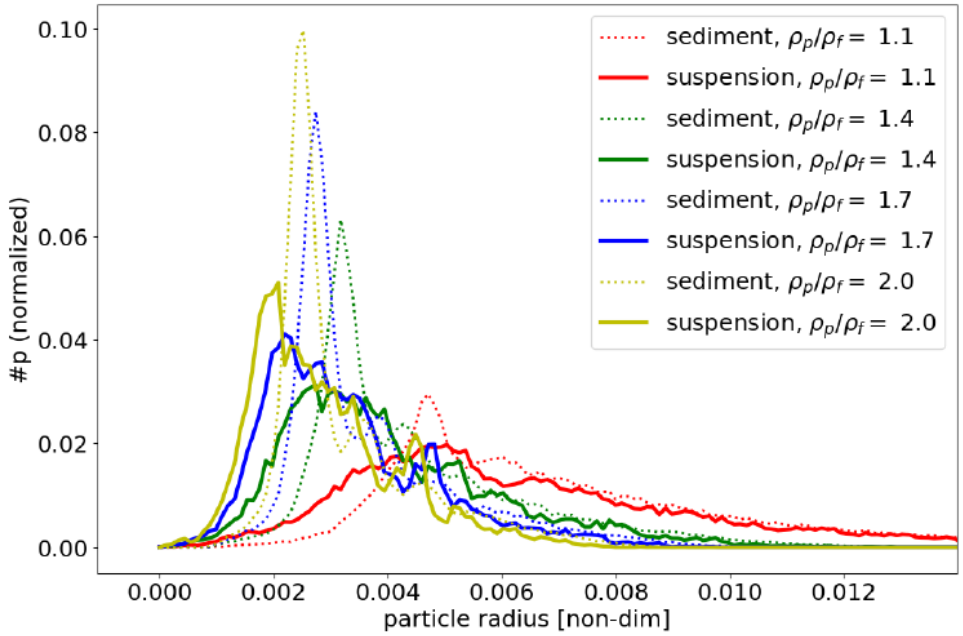






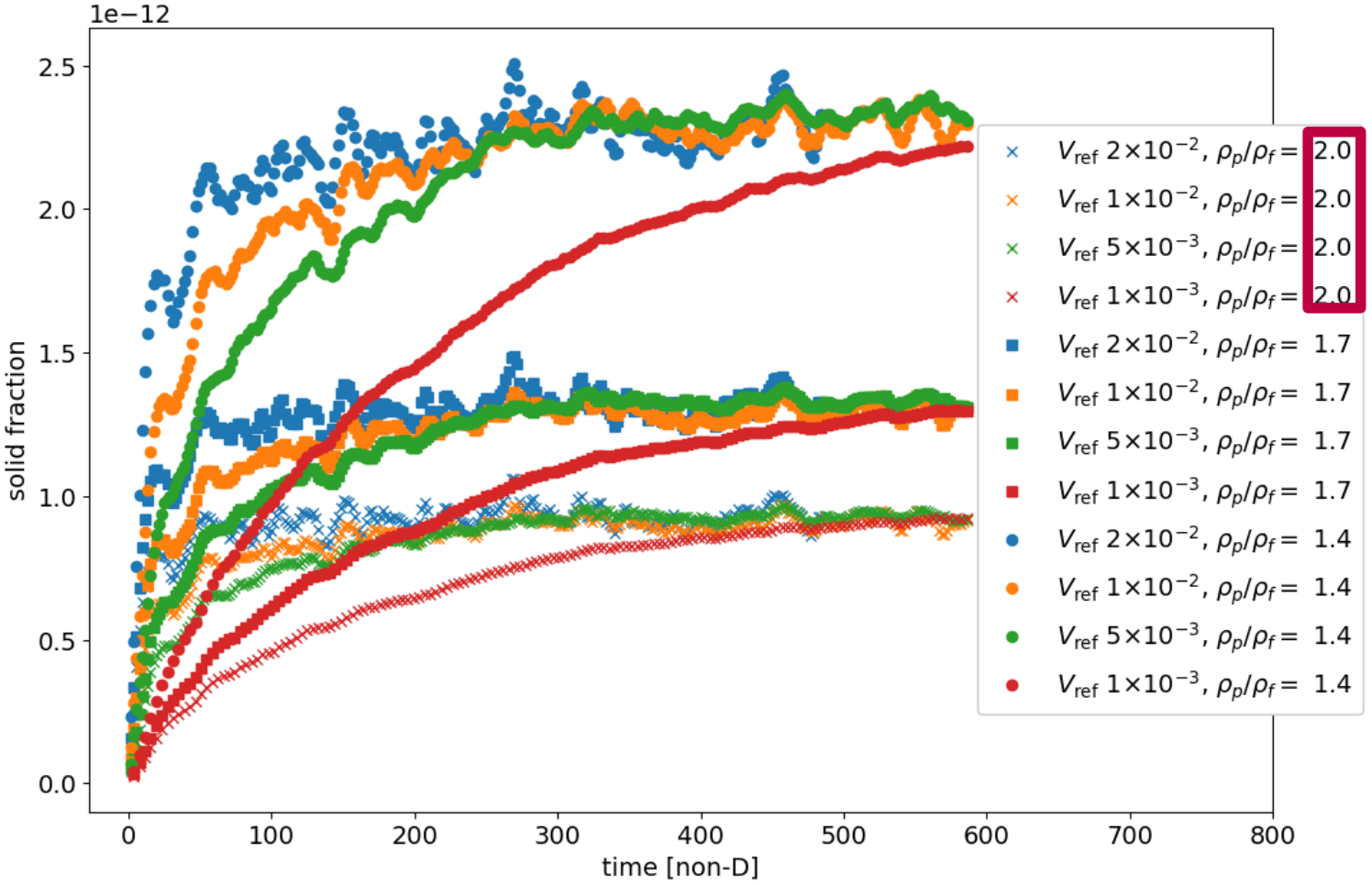
Bulk fluid below the liquidus





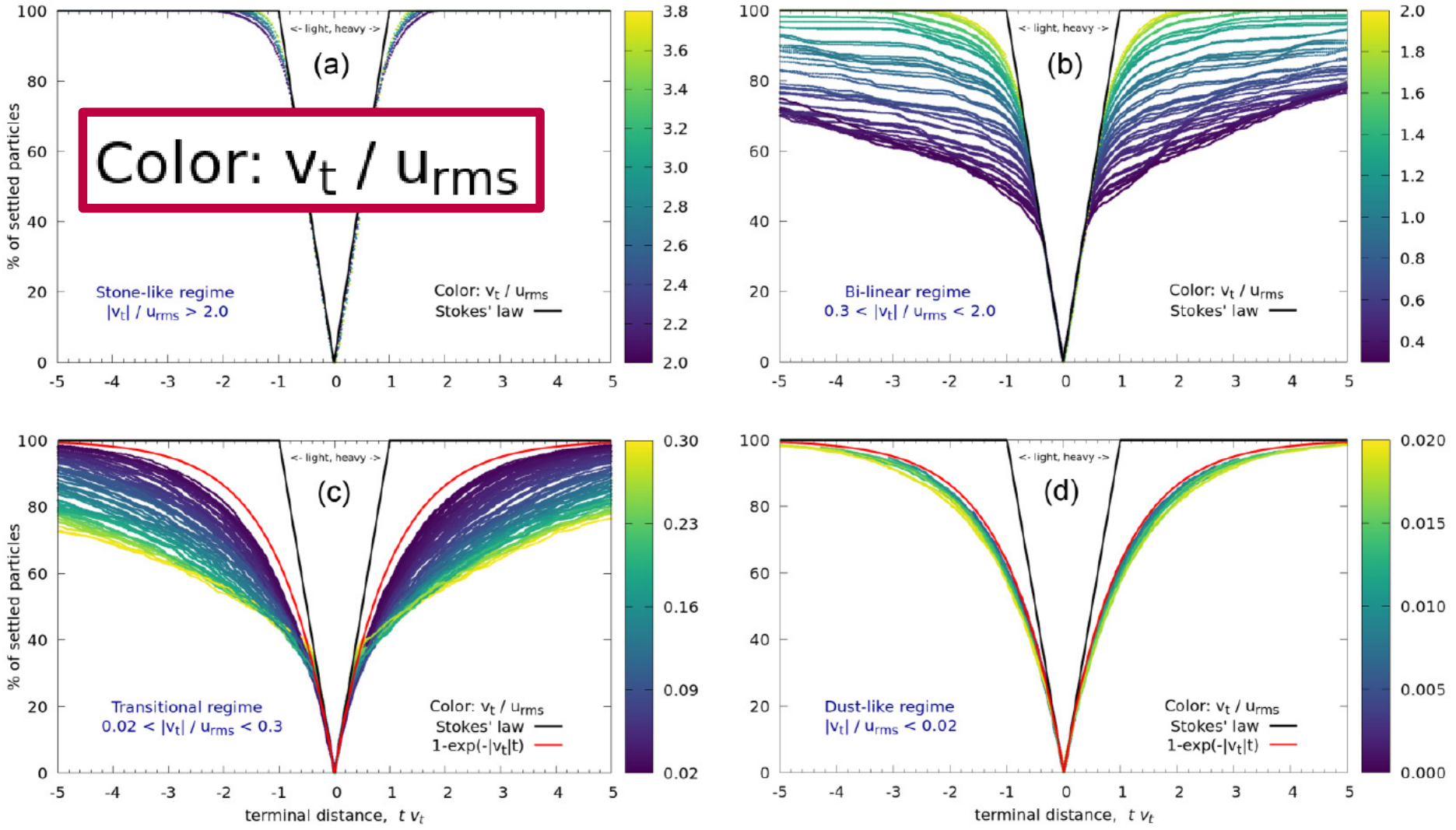
t.d. is the “terminal distance” – key property for dimensionalizing previous results

Solid fraction in suspension



same symbol
= same type

Diluteness does not depend on growth rate!



Color: v_t / u_{rms}

Convective vigor of magma chambers

$$\mathscr{F} = \rho c_p J (T - T_R)^{\frac{4}{3}},$$

$$J \sim 0.16 (g\alpha\kappa^2/\nu)^{\frac{1}{3}}$$

boundary layer thickness:

$$h_b = \rho c_p \kappa (T - T_R)/\mathscr{F} \sim 6.4 Ra^{-\frac{1}{3}} (H - h),$$

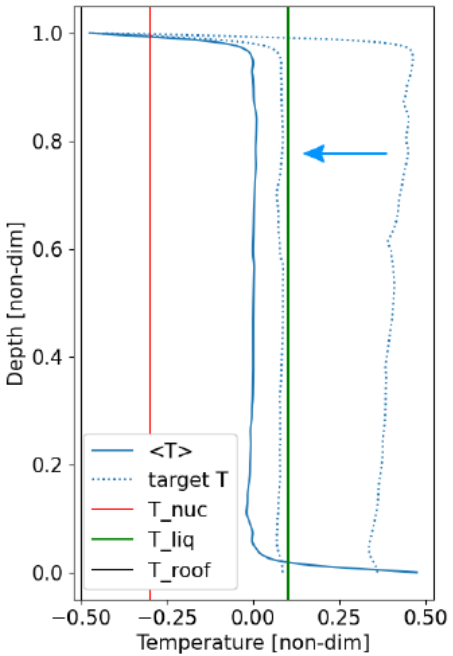
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Sequence of statistically steady states

a) basally heated?

b) internally heated?

   i. uniform release of latent heat?

   ii. locally focused release of latent heat?

Stefan problem

Turcotte & Schubert textbook:

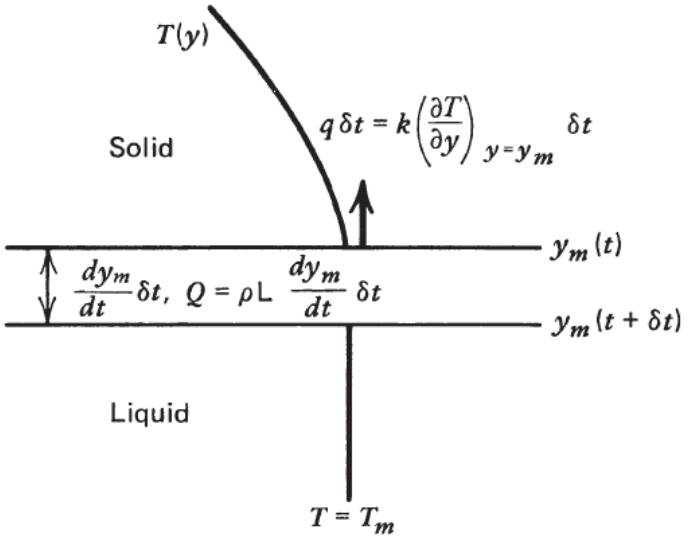


Figure 4.30 Latent heat released at the solidification boundary must be conducted upward through the solidified layer.

$$\frac{L\sqrt{\pi}}{c(T_m - T_0)} = \frac{e^{-\lambda_1^2}}{\lambda_1 \operatorname{erf} \lambda_1},$$

solution for solid layer thickness

Stefan problem

Turcotte & Schubert textbook:

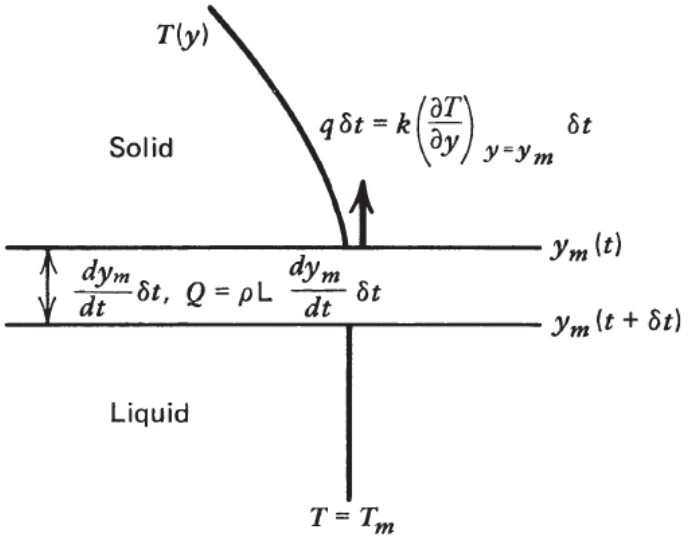
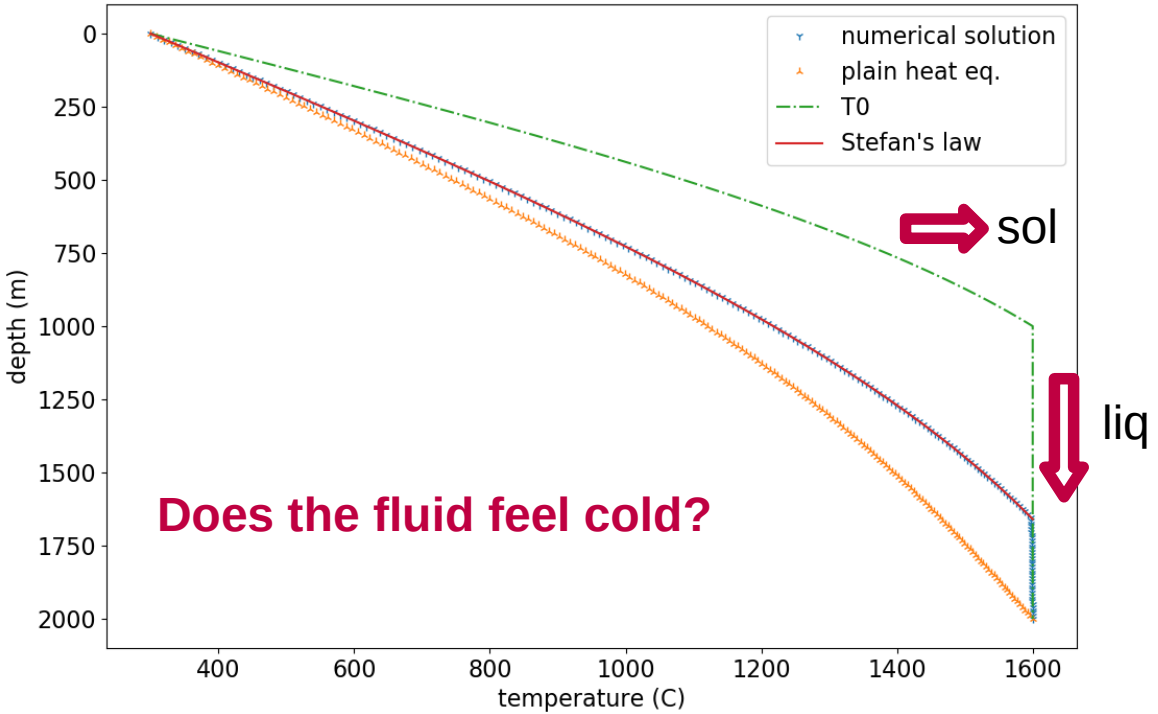


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Bruce D. Marsh (1989):

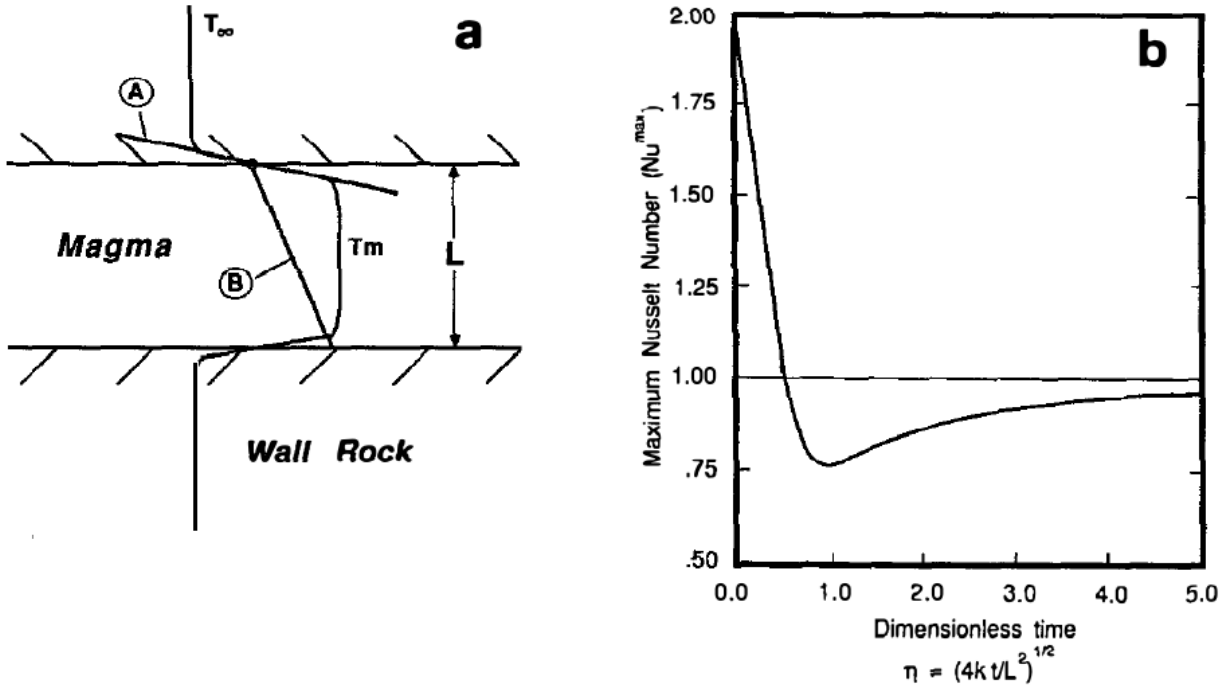


Figure 2 (*a*) The two possible basic thermal states commonly used in defining heat transfer from a magmatic layer of thickness *L*. The steep gradient A is associated with time-dependent conductive cooling, whereas the much less strong gradient B is associated with time-invariant conductive cooling. It is the ratio A/B that is estimated by Equation (12). (*b*) The variation of Nu^{max} with time (dimensionless). This is the rate of heat transfer from a thermally well-mixed, buried sheet relative to the same stagnant sheet. For all times greater than 0.5, the stagnant body cools more rapidly.

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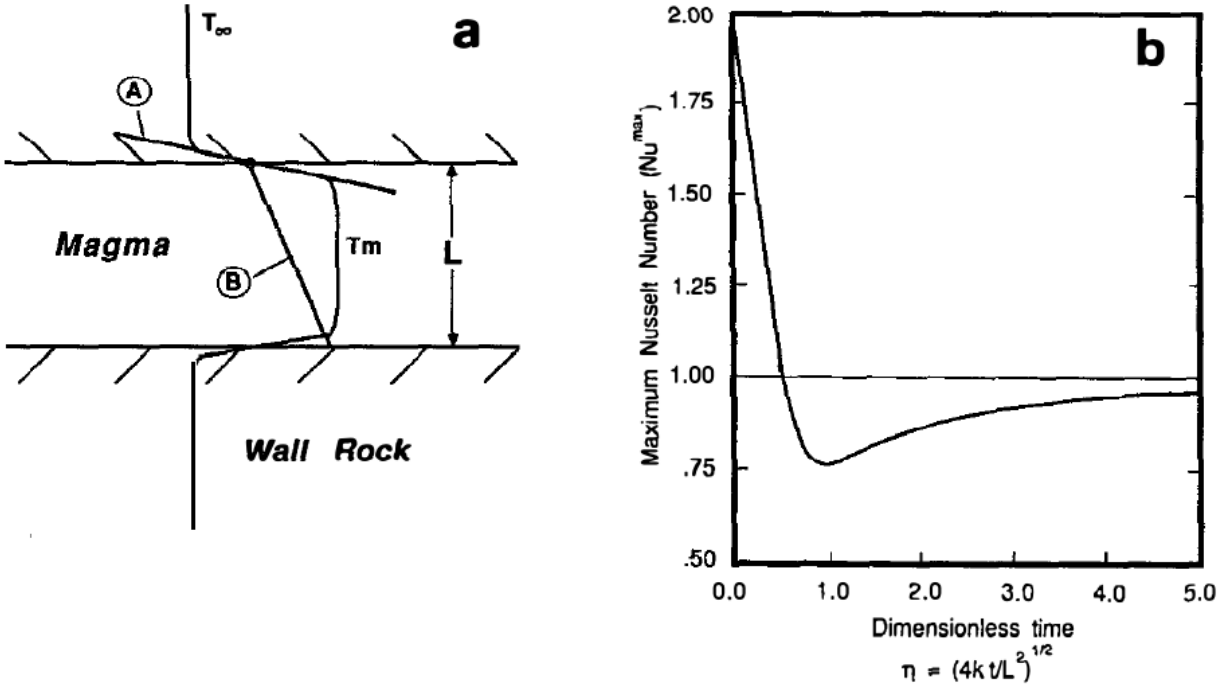


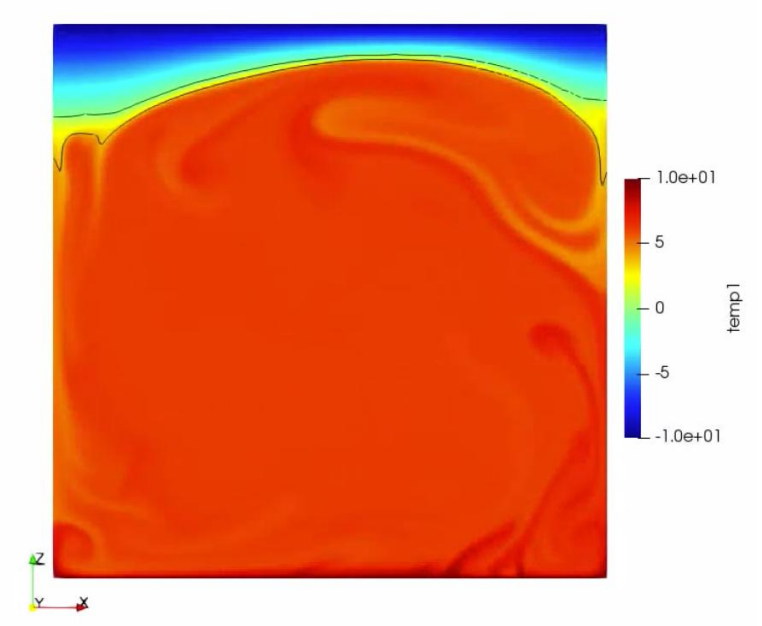
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Valid argument?

Convective vigor of magma chambers

How the growth of ice depends on the fluid dynamics underneath https://doi.org/10.1073/pnas.2012870118

Ziqi Wang, Enrico Calzavarini, Chao Sun, and Federico Toschi



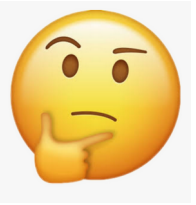
time scale of ice thickening

vs

time scale of fluid flow

Bottom boundary conditions? Fluid T?

Summary:



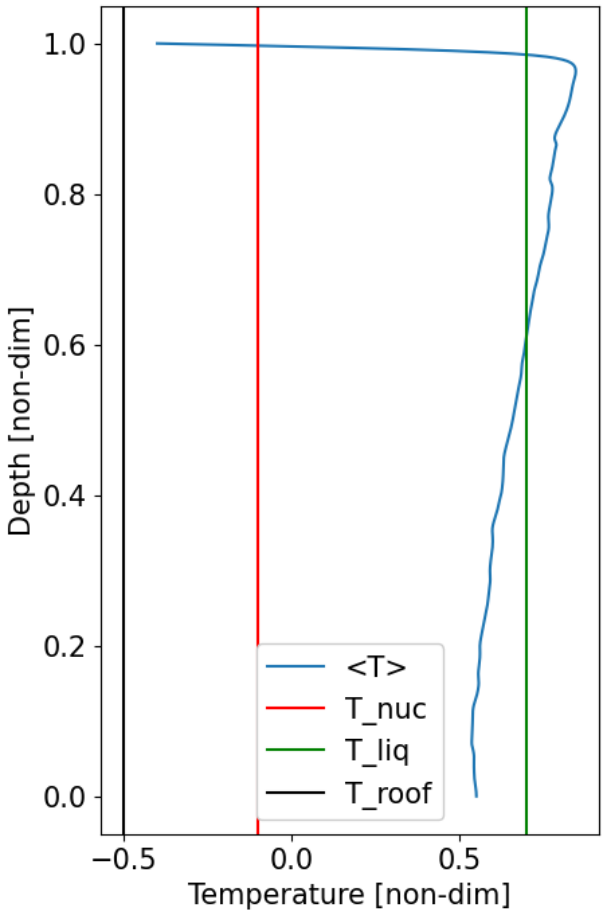
Boundary conditions

Top boundary. The top boundary of a magma chamber should be in principle described by a linear combination of Dirichlet and Neumann BCs (Robin BC) whose relative importance is determined by the Biot number Bi (ratio of thermal resistance of the magma to thermal resistance of the host rock). For most magmatic systems, Bi << 1, which means that in the end the host rock largely acts as a bottleneck for the heat loss.

Bottom boundary. For bottom boundary the situation is different because of the accumulation of solid sediment. The sediment is at the solidus temperature, i.e. it is at constant temperature and thus reducing the basal heat flux. For relatively fast sediment thickening zero bottom heat flux can be assumed.

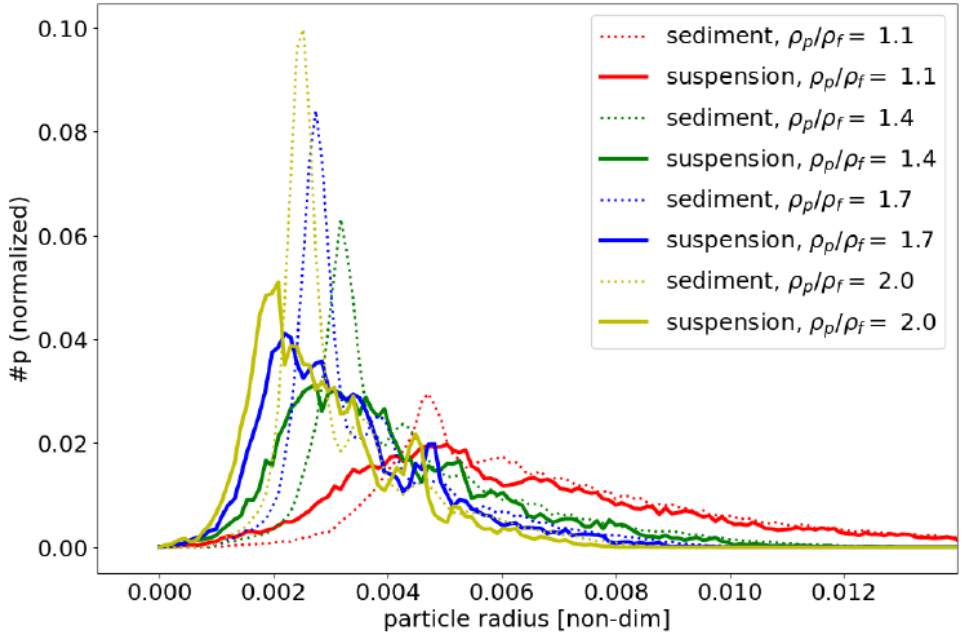
Plan to proceed

- Use the old $Ra_{reference}$ » Ra transformation using the initial temperature contrast as ΔT
- Find statistically steady state
- Investigate its dependence on Ra and N_{ref}
- Analyze characteristic time scales
- Analyze sediment statistics



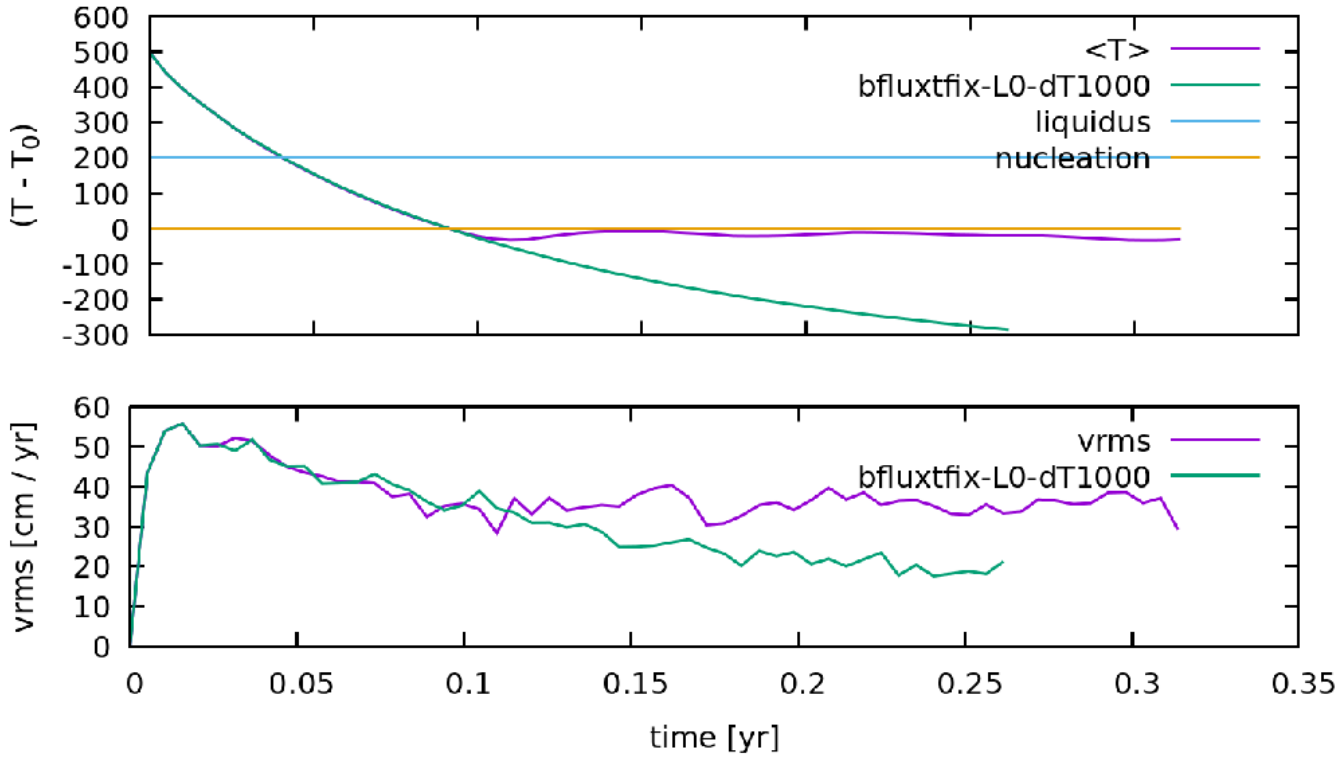
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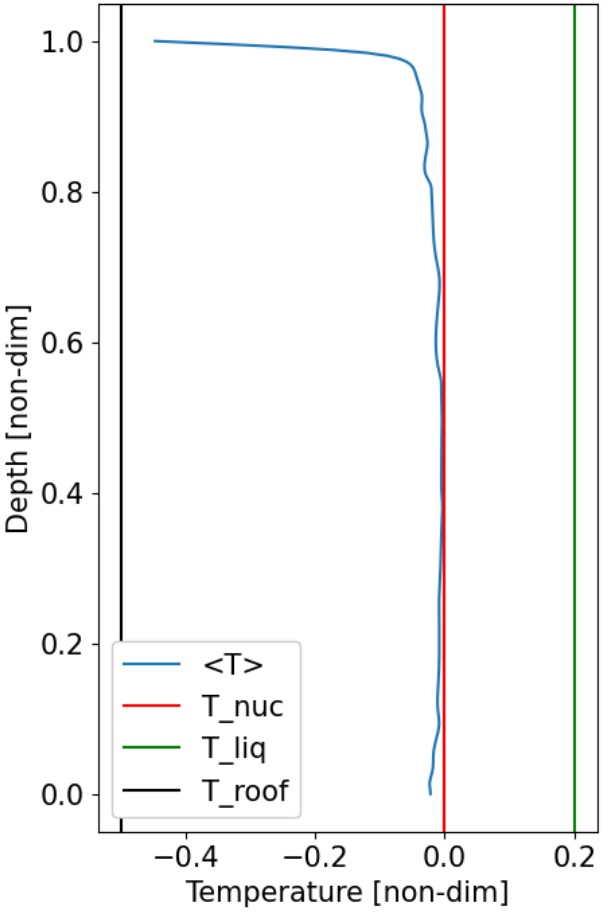
- Use the old $Ra_{reference}$ » Ra transformation using the initial temperature contrast as ΔT
- Find statistically steady state
- Investigate its dependence on Ra and N_{ref}
- Analyze characteristic time scales
- Analyze sediment statistics
- Resolve the small particle mystery (again:)
- Constrain kinetic laws



Thank You

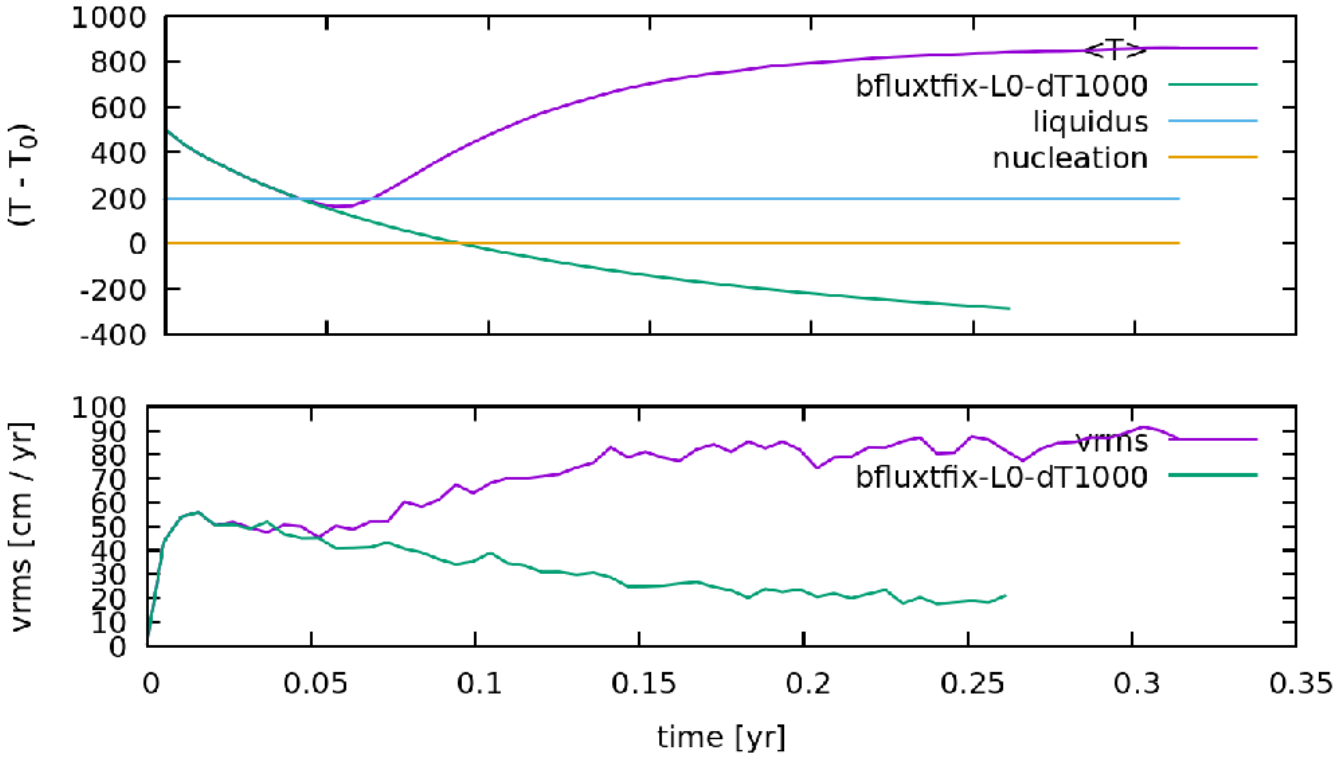
Localized release of latent heat

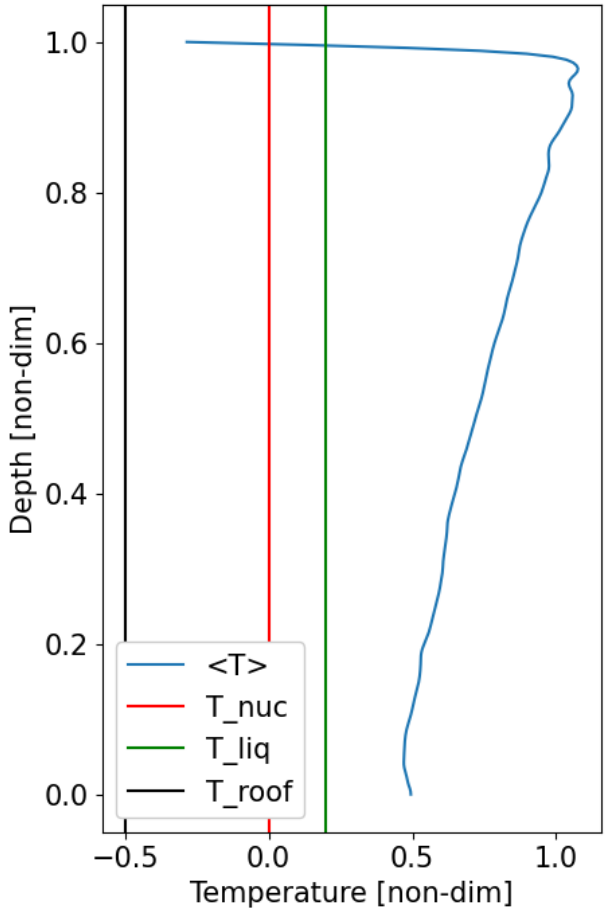




Thank You

Uniform release of latent heat





Numerical model

NVS fluid:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p + \sqrt{\frac{Pr}{Ra}}\nabla^2\mathbf{u} + \theta\,\hat{\mathbf{z}} \\ \nabla\cdot\mathbf{u} &= 0 \\ \partial_\tau \theta + (\mathbf{u}\cdot\nabla)\theta &= \frac{1}{\sqrt{PrRa}}\nabla^2\theta, \end{aligned}$$

Particle model: Simplified Maxey-Riley equation (Basset-Boussinesq-Oseen model)

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \beta\frac{D\mathbf{u}}{Dt} + \frac{1}{St}(\mathbf{u}-\mathbf{v}) + \Lambda\,\hat{\mathbf{z}},$$

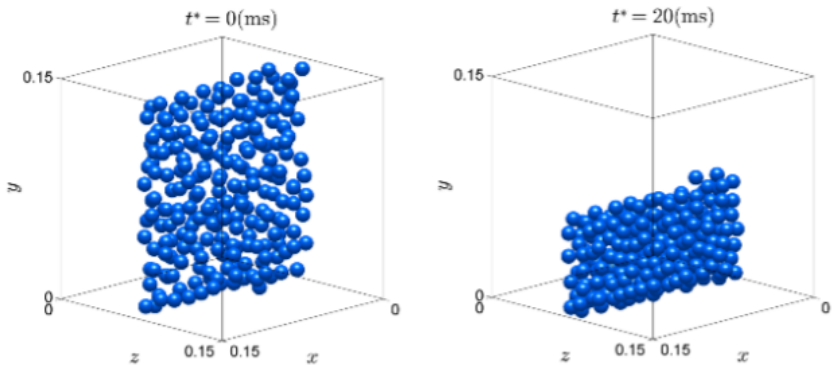
$$\beta = \frac{3\rho_\mathrm{f}}{\rho_\mathrm{f}+2\rho_\mathrm{p}}; \quad St = \frac{r_\mathrm{c}^2\sqrt{\alpha g\Delta T}}{3\nu\beta\sqrt{H}}; \quad \Lambda = \frac{\beta-1}{\alpha\Delta T}.$$

Martin & Nokes model

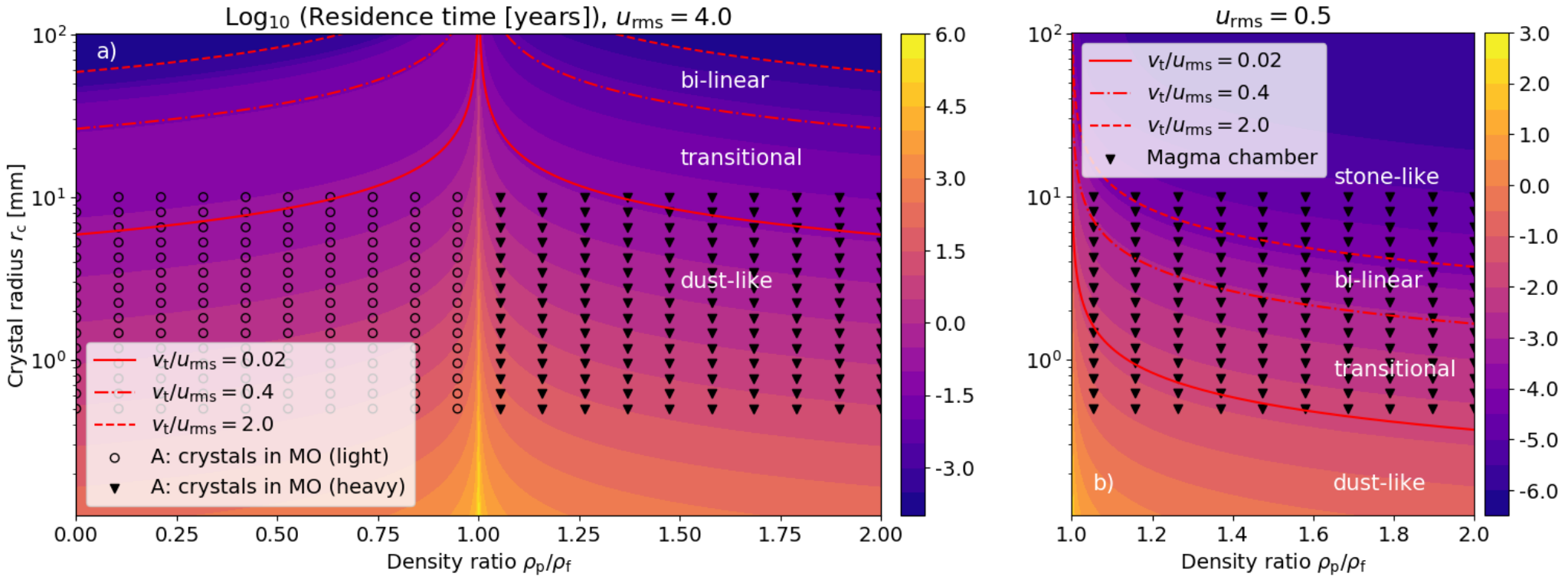
$$\mathrm{M\&N(1989)}: \frac{\mathrm{d}N}{\mathrm{d}t} = -A\, v_\mathrm{t}\, c(0), \quad c(0) = N/AH \quad \Rightarrow \quad N(t) = N_0 \exp\left(\frac{-v_\mathrm{t}\, t}{H}\right)$$

Boundary layer: settling front

Bulk: perfect mixing



Magma oceans and chambers: fractional crystallization



Expected solid fraction

To have a rough idea of how many particles we actually expect to exist in our system, let us convert solidification rate of 1 km of sediment per year to a concentration of free-falling particles with 1 mm radius. The Stokes velocity of particles 10 percent heavier than the fluid is

$$v_\mathrm{t} = \frac{2}{9}\,\frac{\rho_\mathrm{f}-\rho_\mathrm{p}}{\nu\rho_\mathrm{f}}\,r_\mathrm{p}^2\,g = \frac{2}{9}\,\frac{0.1}{5\times10^{-6}}\,(0.001)^2\,9.8\mathrm{m/s} \approx 0.04\mathrm{m/s} \approx 10^3\,\mathrm{km/yr}, \tag{10}$$

which means that the volumetric concentration of such particles must be $\sim 10^{-3}$. For a 2D simulation with aspect ratio 2 and $H = 2890$ km this is $\sim 5\times10^{11}$ particles.

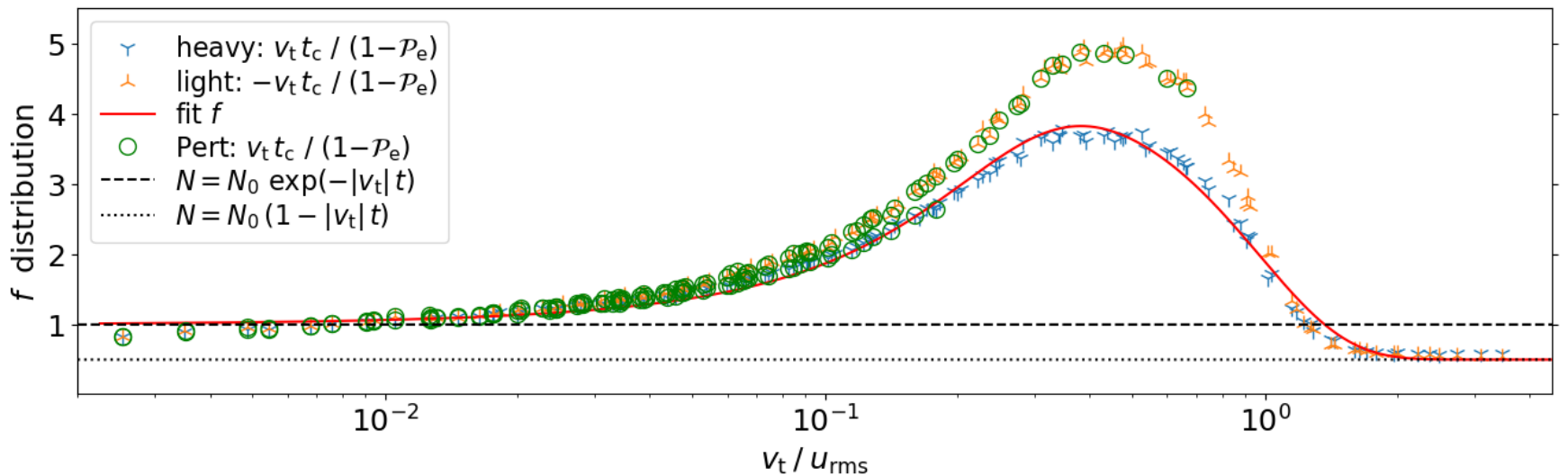
Perturbation theory

Maxey – Riley eq.:

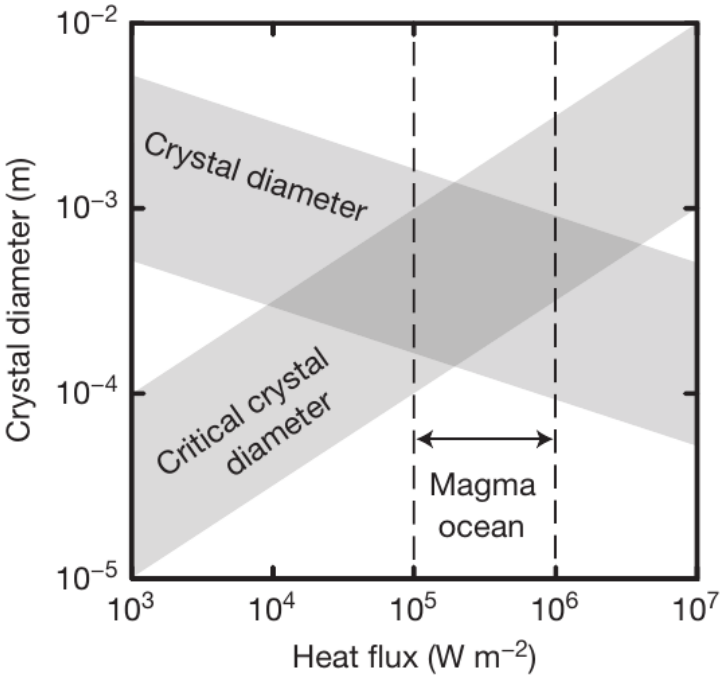
$$\frac{d\mathbf{v}}{dt} = \beta\frac{D\mathbf{u}}{Dt} + \frac{1}{\tau_D}(\mathbf{u}-\mathbf{v}) + (1-\beta)\mathbf{g}$$

First-order solution:

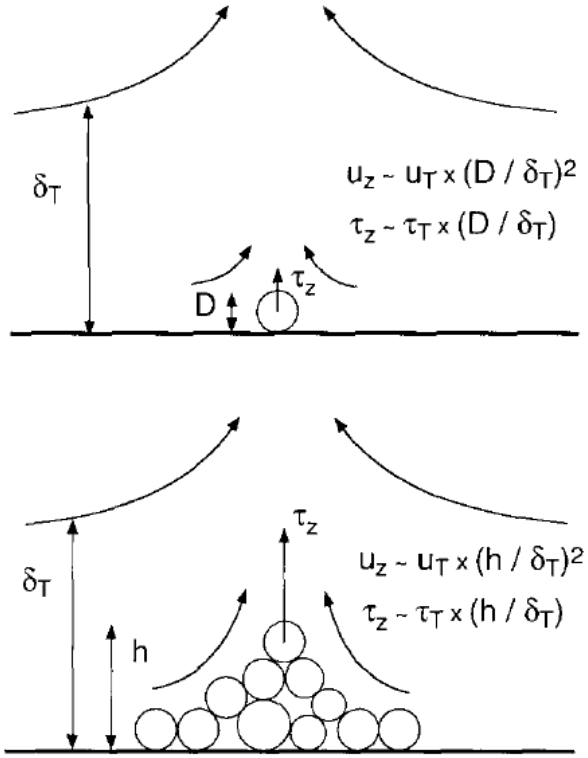
$$\mathbf{v} \simeq \mathbf{u} + \tau_D(1-\beta)\left(\mathbf{g} - \frac{D\mathbf{u}}{Dt}\right) + \mathcal{O}(\tau_D^2)$$

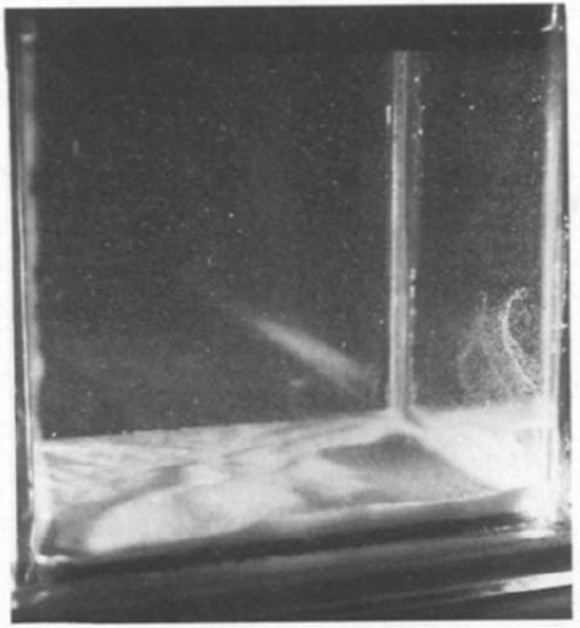


Re-entrainment



Solomatov, ToG 2015





Series of papers by Solomatov, Stevenson, and Olson in 1993