

Selfgravitating viscoelastic continua towards formation of early planets.

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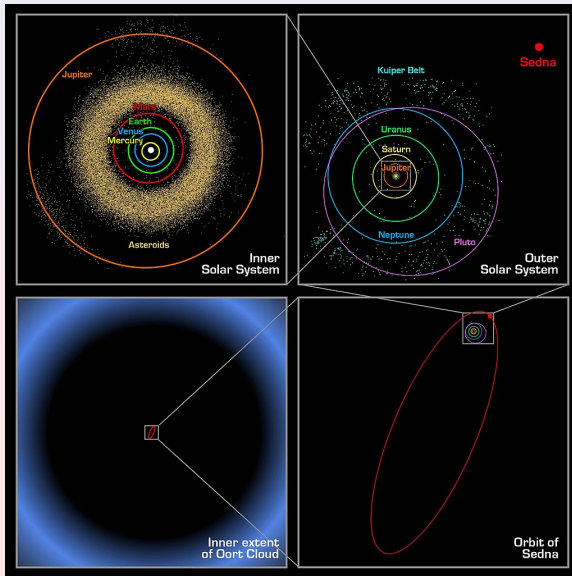
Motivation:

- accretion of terrestrial-type planets from the solar nebula
(~ 4.6 billion years ago during about 100 My)
- differentiation of a silicate mantle and a metallic core (next ~ 1 Gy).

Mechanisms to be considered:

- continuum mechanics of viscoelastic fluids (+ inertial effects)
- self-gravitation
- thermodynamics (heat production during accretion and differentiation)
- transition from a dilute dust to a dense viscoelastic terrestrial-type planet
- open systems (incoming volume, mass, momentum, energy)
- two-component continuum (lighter silicates + heavier metals)

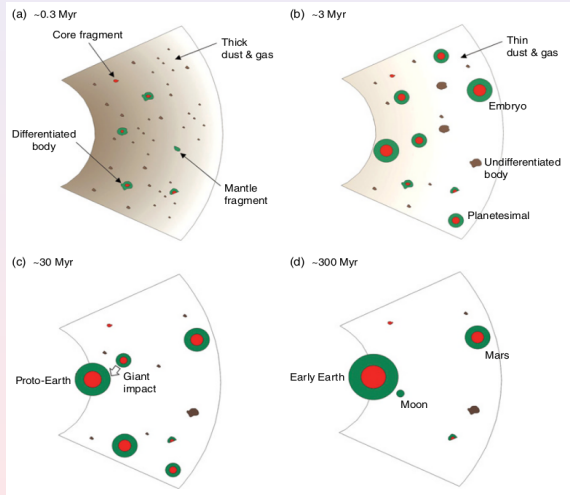
The **Solar system** in our times about 4,6 Gy after the beginning of its formation:



SOURCE:
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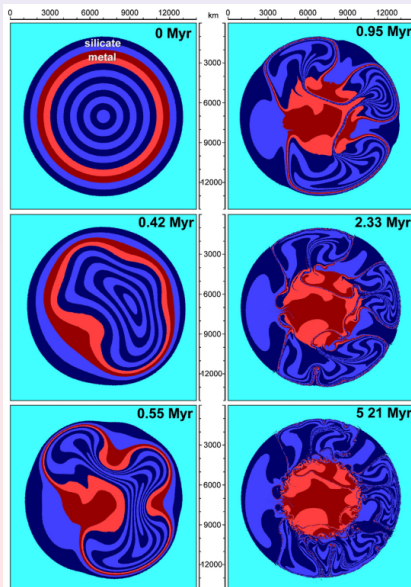
JOHANN E. BODE's law:
the orbit of n^{th} -planet:
 $\text{radius} \doteq 0.4 + 0.3 \times 2^n \text{ AU}$

A schematic illustration of beginning of planetary system formation:



F.NIMMO & T.KLEINE: Early Differentiation and Core Formation: Processes and Timescales. Ch.5 in: *The Early Earth Accretion and Differentiation*. (Eds. J.Badro & M.Walter), J.Wiley, 2015.

An illustration of the core-mantle formation (a 2D numerical test):



T.V. GERYA & D.A. YUEN:
Robust characteristics method
for modelling multiphase
visco-elasto-plastic thermo-
mechanical problems. *Physics
of the Earth and Planetary
Interiors* 163 (2007) 83–105

Content of the talk:

- 1 Finite-strain kinematics
- 2 Eulerian mechanics of solids and fluids
 - Kelvin-Voigt materials in Eulerian description
 - Viscoelastic fluids
 - Selfgravitation
- 3 Further extensions
 - Thermodynamics
 - Open thermodynamical system
 - Multicomponent model

Finite-strain kinematics

Deformation $\mathbf{y} : \Omega \rightarrow \mathbb{R}^d$ as a mapping from a reference configuration $\Omega \subset \mathbb{R}^d$ into a physical space $\mathbb{R}^d$.

Notation: $\mathbf{X} \in \Omega$ the “Lagrangian” space variable in the reference configuration.
 $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ the “Eulerian” physical-space variable.

The basic geometrical object is the (referential) deformation gradient $\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{y}$,
with deformations $\mathbf{x} = \mathbf{y}(t, \mathbf{X})$ evolving in time (sometimes called motion)
the (referential) velocity $\mathbf{v} = \frac{\partial}{\partial t} \mathbf{y}$.
the inverse of a motion $\boldsymbol{\xi}(t) : \mathbf{x} \mapsto \mathbf{y}^{-1}(t, \mathbf{x})$, called
the return (or reference) mapping.

Then we consider the actual (i.e. Eulerian) deformation gradient $\mathbf{F} = \mathbf{F} \circ \boldsymbol{\xi}$
and the actual velocity $\mathbf{v} = \mathbf{v} \circ \boldsymbol{\xi}$.

The return mapping $\boldsymbol{\xi}$ satisfies the transport equation

$$\dot{\boldsymbol{\xi}} = \mathbf{0}.$$

Notation:

$$(\cdot)^* = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \text{ for the convective time derivative}$$

(with a shorthand notation $\nabla := \nabla_{\mathbf{x}}$)

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The velocity gradient $\nabla \mathbf{v} = \nabla_{\mathbf{x}} \mathbf{v} \nabla_{\mathbf{x}} \mathbf{X} = \dot{\mathbf{F}} \mathbf{F}^{-1}$,
(the chain-rule calculus and $\mathbf{F}^{-1} = (\nabla_{\mathbf{x}} \mathbf{x})^{-1} = \nabla_{\mathbf{x}} \mathbf{X}$).

The transport-and-evolution equation for the deformation gradient as

$$\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}.$$

the evolution-and-transport equation for the Jacobian $J = \det \mathbf{F}$ as

$$\dot{J} = \text{Cof} \mathbf{F} : \dot{\mathbf{F}} = J \mathbf{F}^{-\top} : \dot{\mathbf{F}} = J \mathbb{I} : \dot{\mathbf{F}} \mathbf{F}^{-1} = J \mathbb{I} : \nabla \mathbf{v} = J \text{div } \mathbf{v},$$

where $\mathbb{I}$ denotes the unit matrix.

A distortion $\mathbf{A} = \nabla \xi = \mathbf{F}^{-1}$ and together with its determinant $\det \mathbf{A} = 1/J$ satisfy the transport-and-evolution equations

$$\dot{\mathbf{A}} = -\mathbf{A}(\nabla \mathbf{v}) \quad \text{and} \quad \frac{\dot{\det \mathbf{A}}}{\det \mathbf{A}} = -(\det \mathbf{A}) \text{div } \mathbf{v}.$$

equivalently $\frac{\partial}{\partial t} \det \mathbf{A} + \text{div}(\mathbf{v} \det \mathbf{A}) = 0$, i.e. $\frac{1}{J}$ is transported as an extensive variable.

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Reconstruction of an underlying deformation $\mathbf{y}$:

Is ξ bijective? Not automatically!

Local non-interpenetrability $\det \mathbf{F} > 0$ and
fixing of (invertible) boundary conditions on $\mathbf{y}$

$\Rightarrow$ global injectivity of $\xi(t)$, i.e. $\mathbf{y}(t) = \xi^{-1}(t)$ exists.

$\Rightarrow \mathbf{F} = \nabla \mathbf{y}$.

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Kelvin-Voigt materials in Eulerian description

Referential mass density: $\varrho_R = \varrho_R(\mathbf{X})$, actual mass density: $\varrho = \varrho(t, \mathbf{x})$,

The continuity equation: $\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0$, $\varrho|_{t=0} = \varrho_R \Rightarrow \varrho = \frac{\varrho_R \circ \xi}{\det \mathbf{F}}$.

Momentum equation: $\varrho \dot{\mathbf{v}} = \operatorname{div}(\mathbf{S}_1 + \mathbf{S}_2) + \mathbf{f}$ (e.g. gravitational $\mathbf{f}(\mathbf{x}) = \varrho(\mathbf{x})\mathbf{g}$)
or equivalently $\frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div}(\mathbf{S}_1 + \mathbf{S}_2) + \mathbf{f}$.

3 options for the stored energy: $\varphi = \begin{cases} \varphi(\mathbf{X}, \mathbf{F}) & \text{referential in J/m}^3 = \text{Pa}, \\ \varphi(\mathbf{X}, \mathbf{F}) & \text{referential in J/kg}, \\ \varphi(\mathbf{x}, \mathbf{F}) & \text{actual in Pa.} \end{cases}$

Corresponding conservative Cauchy stress: $\mathbf{S}_1 = \begin{cases} \varphi'_F(\xi(\mathbf{x}), \mathbf{F}) \mathbf{F}^\top / \det \mathbf{F}, \\ \varrho(\mathbf{x}) \varphi'_F(\xi(\mathbf{x}), \mathbf{F}) \mathbf{F}^\top, \\ \varphi'_F(\mathbf{x}, \mathbf{F}) \mathbf{F}^\top + \varphi(\mathbf{x}, \mathbf{F}) \mathbb{I}. \end{cases}$

Dissipative Cauchy stress: $\mathbf{S}_2 = \mathbb{D} \mathbf{e}(\mathbf{v})$ with a 4th-order positive-definite viscous-moduli tensor $\mathbb{D}$.

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Overall stored energy: $\Phi(\mathbf{F}) = \int_{\Omega} \begin{cases} \varphi(\boldsymbol{\xi}(\mathbf{x}), \mathbf{F}(\mathbf{x})) / \det \mathbf{F}(\mathbf{x}) \\ \varrho(\mathbf{x}) \varphi(\boldsymbol{\xi}(\mathbf{x}), \mathbf{F}(\mathbf{x})) \\ \varphi(\mathbf{x}, \mathbf{F}(\mathbf{x})) \end{cases} d\mathbf{x}.$

The **dissipative gradient** extension of the Kelvin-Voigt system:

$$\dot{\varrho} = -(\operatorname{div} \mathbf{v}) \varrho,$$

$$\varrho \dot{\mathbf{v}} - \operatorname{div} \left(\mathbf{S}_1 + \mathbb{D} \mathbf{e}(\mathbf{v}) - \operatorname{div} \mathcal{H}(\nabla^2 \mathbf{v}) \right) = \varrho \mathbf{g},$$

$$\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}.$$

Hyper-viscosity term with $\mathcal{H}(\nabla^2 \mathbf{v}) = \nu |\nabla^2 \mathbf{v}|^{p-2} \nabla^2 \mathbf{v}$.

Multipolar fluids: J. Nečas & collaborators (1989-...) F. Bloom, A. Novotný,
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later also E. Fried and M.E. Gurtin (2005) as 2nd-gradient fluids.

$\Rightarrow$ $\mathbf{v}$ is “nicely” regular (Lipschitz in space integrably in time)

$\Rightarrow$ transport by $\mathbf{v}$ of $\mathbf{F}$ by $\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}$ and also other transports (here ϱ)
replicates regularity of initial conditions.

Note: the normal dispersion of wave velocity is linear analog

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The **energy dissipation balance** (test by $\mathbf{v}$ and by $\varphi(\mathbf{F})$, + B.C. prescribed):

$$\frac{d}{dt} \left(\int_{\Omega} \underbrace{\frac{\rho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} d\mathbf{x} + \underbrace{\phi(\mathbf{F})}_{\text{stored energy}} \right) + \int_{\Omega} \underbrace{\mathbb{D}\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v}) + \nu |\nabla^2 \mathbf{v}|^p}_{\text{dissipation rate due to Stokes viscosity}} d\mathbf{x} = \int_{\Omega} \underbrace{\rho \mathbf{g} \cdot \mathbf{v}}_{\text{power of gravity field}} d\mathbf{x}.$$

Some notes about **analysis**

Extremely nonlinear $\Rightarrow$ time discretization (Rothe) does not work.

A nonlinear test of $\dot{\mathbf{F}} = (\nabla \mathbf{v})\mathbf{F}$ by $\varphi(\mathbf{F})$ and possibly also
a nonlinear test of $\dot{\varrho} = -(\operatorname{div} \mathbf{v})\varrho$ by $|\mathbf{v}|^2/2$

$\Rightarrow$ Galerkin discretization does not work.

$\Rightarrow$ a Galerkin “semidiscretization”:

the momentum equation (to be tested by $\mathbf{v}$) discretized conformally

the linear transport-and-evolution equations $\dot{\mathbf{F}} = (\nabla \mathbf{v})\mathbf{F}$, $\dot{\varrho} = -(\operatorname{div} \mathbf{v})\varrho$,

$$\left(\frac{\dot{1}}{\det \mathbf{F}} \right) = -\frac{\operatorname{div} \mathbf{v}}{\det \mathbf{F}}, \text{ and } \left(\frac{\dot{1}}{\varrho} \right) = \frac{\operatorname{div} \mathbf{v}}{\varrho}, \text{ or also } \dot{\xi} = \mathbf{0}$$

kept continuous while proving their unique regular solutions for a given

$$\mathbf{v} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \subset L^1(I; W^{1,\infty}(\Omega; \mathbb{R}^d)) \text{ for } p > d.$$

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$$\frac{d}{dt} \left(\underbrace{\int_{\Omega} \frac{\rho}{2} |\mathbf{v}|^2 dx}_{\text{kinetic energy}} + \underbrace{\phi(\mathbf{F})}_{\text{stored energy}} \right) + \int_{\Omega} \underbrace{\mathbb{D}\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v}) + \nu |\nabla^2 \mathbf{v}|^p}_{\text{dissipation rate due to Stokes viscosity}} dx = \int_{\Omega} \underbrace{\rho \mathbf{g} \cdot \mathbf{v}}_{\text{power of gravity field}} .$$

Some notes about **analysis**

Extremely nonlinear $\Rightarrow$ time discretization (Rothe) does not work.

A nonlinear test of $\dot{\mathbf{F}} = (\nabla \mathbf{v})\mathbf{F}$ by $\varphi(\mathbf{F})$ and possibly also
a nonlinear test of $\dot{\varrho} = -(\operatorname{div} \mathbf{v})\varrho$ by $|\mathbf{v}|^2/2$

$\Rightarrow$ Galerkin discretization does not work.

$\Rightarrow$ a **Galerkin “semidiscretization”**:

the momentum equation (to be tested by $\mathbf{v}$) discretized conformally

the linear transport-and-evolution equations $\dot{\mathbf{F}} = (\nabla \mathbf{v})\mathbf{F}$, $\dot{\varrho} = -(\operatorname{div} \mathbf{v})\varrho$,

$$\overline{\left(\frac{1}{\det \mathbf{F}}\right)} = -\frac{\operatorname{div} \mathbf{v}}{\det \mathbf{F}}, \quad \text{and} \quad \overline{\left(\frac{1}{\varrho}\right)} = \frac{\operatorname{div} \mathbf{v}}{\varrho}, \quad \text{or also} \quad \dot{\xi} = \mathbf{0}$$

kept continuous while proving their unique regular solutions for a given

$$\mathbf{v} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \subset L^1(I; W^{1,\infty}(\Omega; \mathbb{R}^d)) \text{ for } p > d.$$

A particular case: viscoelastic fluids

Fluids = shear elastic response vanishes

φ depends only on the isochoric part $(\det \mathbf{F})^{\mathbb{I}}$ of $\mathbf{F}$ and
 completely ignores the deviatoric part $\mathbf{F}/\det \mathbf{F}$.

$\Rightarrow \varphi(\mathbf{F}) = \phi(J)$ with $J = \det \mathbf{F}$ and some $\phi : (0, +\infty) \rightarrow (0, +\infty)$.
 (For simplicity: a spatially homogeneous material.)

Liquids do not withstand (too much big) negative pressures (tension)

$\Rightarrow \phi(\cdot)$ is not coercive for $J \rightarrow +\infty$.

Gases do not withstand negative pressures (tension) at all and

$\Rightarrow p = 0 \quad \& \quad \varrho = 0$ for $J \rightarrow +\infty$.

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1st option: a referential ϕ in J/m^3 :

The calculus $\det'(\cdot) = \text{Cof}(\cdot)$ and the algebra $F^{-1} = \text{Cof} F^\top / \det F$,
 $\Rightarrow$ the conservative part of the Cauchy stress then reduces to

$$\mathbf{S}_1 = \frac{\phi'(\mathbf{F})\mathbf{F}^\top}{\det \mathbf{F}} = \phi'(\det \mathbf{F}) \frac{\det'(\mathbf{F})\mathbf{F}^\top}{\det \mathbf{F}} = \phi'(\det \mathbf{F}) \frac{(\text{Cof } \mathbf{F})\mathbf{F}^\top}{\det \mathbf{F}} = \phi'(J)\mathbb{I}.$$

Replacing $\dot{\varrho} = -\varrho \text{div } \mathbf{v}$ with initial conditions $\varrho|_{t=0} = \rho_R$ and $J|_{t=0} = 1$:

$\Rightarrow$ The “solid” 3+3 system reduces to the system of 3+1 equations:

$$\dot{\mathbf{v}} = \frac{J}{\rho_R} \left(\text{div}(\mathbb{D}\mathbf{e}(\mathbf{v}) + \nabla p) + \mathbf{g} \right) \quad \text{with} \quad p = -\phi'(J), \quad \text{and}$$

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The conservative Cauchy stress:

$$\begin{aligned}\mathbf{S}_1 &= \varphi'(\mathbf{F})\mathbf{F}^\top + \varphi(\mathbf{F})\mathbb{I} = [\phi(\det \mathbf{F})]'_{\mathbf{F}} \mathbf{F}^\top + \phi(\det \mathbf{F})\mathbb{I} \\ &= \phi'(J)(\text{Cof } \mathbf{F})\mathbf{F}^\top + \phi(J)\mathbb{I} = (\phi'(J)J + \phi(J))\mathbb{I}.\end{aligned}$$

Using also the calculus $(\mathbf{v} \cdot \nabla)\mathbf{v} = \text{rot } \mathbf{v} \times \mathbf{v} + \frac{1}{2}\nabla|\mathbf{v}|^2$, the system:

$$\begin{aligned}\dot{\varrho} &= -\varrho \text{div } \mathbf{v} \quad \text{and} \quad \dot{J} = J \text{div } \mathbf{v}, \\ \varrho \dot{\mathbf{v}} &+ \underbrace{\varrho \text{rot } \mathbf{v} \times \mathbf{v}}_{\text{Lamb force}} - \text{div} \left(\mathbb{D}\mathbf{e}(\mathbf{v}) - \underbrace{\left(\frac{\varrho}{2}|\mathbf{v}|^2 + \phi'(J)J + \phi(J) \right)}_{\text{"Bernoulli pressure"}} \mathbb{I} \right) = \varrho \mathbf{g}.\end{aligned}$$

$\mathbb{D} \sim 0$ and $\mathbf{g} \sim 0$ and the flow is steady and makes $\text{rot } \mathbf{v}$ orthogonal to $\mathbf{v}$,

$\Rightarrow$ the Bernoulli principle is revealed:

$$\frac{\varrho}{2}|\mathbf{v}|^2 + \phi'(J)J + \phi(J) \sim \text{constant}.$$

D. Bernoulli: *Hydrodynamica, sive de viribus et motibus fluidorum commentarii*. 1738.

DOI: 10.3931/erarc-3911 

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Selfgravitation

Gravitation (non-relativistic):

The **POISSON equation**: $\Delta V = G\rho$ on $\mathbb{R}^3$

V gravitational potential,

G normalized gravitational constant $\doteq 4\pi \cdot 6.674 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$.

The usual “boundary” condition $V(\infty) := \lim_{|\mathbf{x}| \rightarrow \infty} V(\mathbf{x}) = 0$.

$\Rightarrow$ an integral formula involving the Green function of the Δ -operator as

$$V(\mathbf{x}) = -G \int_{\Omega} \frac{\rho(\tilde{\mathbf{x}})}{|\mathbf{x} - \tilde{\mathbf{x}}|} d\tilde{\mathbf{x}};$$

$\Rightarrow$ gravitational acceleration:

$$\mathbf{g}(\mathbf{x}) = -G \nabla V(\mathbf{x}) = G \int_{\Omega} \frac{\rho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \mathbf{x})}{|\tilde{\mathbf{x}} - \mathbf{x}|^3} d\tilde{\mathbf{x}}. \quad (\text{NEWTON's law.})$$

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The overall isothermal **system** for selfgravitating fluids, i.e.

the **compressible Navier-Stokes-Poisson system** (1st option: ϕ referential in Pa).

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0, \quad (\text{mass-density continuity equation})$$

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} - \delta'_e(J; \mathbf{e}(\mathbf{v}))) + \nabla p = \varrho \mathbf{g} \quad (\text{momentum equation})$$

$$\text{with } p = -\psi'_J(J), \quad \mathbf{g} = G \int_{\Omega} \frac{\varrho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \cdot)}{|\tilde{\mathbf{x}} - \cdot|^3} d\tilde{\mathbf{x}},$$

$$\frac{\partial J}{\partial t} = (\operatorname{div} \mathbf{v})J - \mathbf{v} \cdot \nabla J. \quad (\text{kinematic "volume equation"})$$

+ the impermeable friction-free boundary: $\mathbf{v} \cdot \mathbf{n} = 0$ and $[\delta'_e(J, \theta; \mathbf{e}(\mathbf{v}))\mathbf{n}]_{\mathbb{T}} = \mathbf{0}$ on Γ
where $[\cdot]_{\mathbb{T}}$ denotes the tangential component of a traction vector.

$$\Rightarrow \text{Mass conservation: } \frac{d}{dt} \int_{\Omega} \varrho d\mathbf{x} = 0.$$

$$\Rightarrow \text{Linear momentum conservation: } \frac{d}{dt} \int_{\Omega} \varrho \mathbf{v} d\mathbf{x} = 0.$$

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Testing the momentum equation by $\mathbf{v}$ and using the calculus:

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{p} \cdot \mathbf{v} \, d\mathbf{x} &= \int_{\Omega} \phi'(J) \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \phi'(J) \frac{\dot{J}}{J} \, d\mathbf{x} = \int_{\Omega} \left(\left[\frac{\phi(J)}{J} \right]' + \frac{\phi(J)}{J^2} \right) \dot{J} \, d\mathbf{x} \\ &= \int_{\Omega} \left[\frac{\phi(J)}{J} \right]' \dot{J} + \frac{\phi(J)}{J} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{\phi(J)}{J} \right) + \nabla \left(\frac{\phi(J)}{J} \right) \cdot \mathbf{v} + \frac{\phi(J)}{J} \operatorname{div} \mathbf{v} \, d\mathbf{x} \\ &= \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Omega} \operatorname{div} \left(\frac{\phi(J)}{J} \mathbf{v} \right) \, d\mathbf{x} = \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Gamma} \frac{\phi(J)}{J} \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} \, dS. \end{aligned}$$

and testing $V = G\Delta^{-1}\varrho$ by $\frac{\partial}{\partial t} V$ and using the calculus:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\nabla V|^2}{2G} \, d\mathbf{x} &= - \int_{\mathbb{R}^3} \frac{\Delta V}{G} \frac{\partial V}{\partial t} \, d\mathbf{x} = - \int_{\Omega} \varrho \frac{\partial V}{\partial t} \, d\mathbf{x} = \int_{\Omega} \frac{\partial \varrho}{\partial t} V \, d\mathbf{x} - \frac{d}{dt} \int_{\Omega} \varrho V \, d\mathbf{x} \\ &= - \int_{\Omega} V \operatorname{div}(\varrho \mathbf{v}) \, d\mathbf{x} - \frac{d}{dt} \int_{\Omega} \varrho V \, d\mathbf{x} = \int_{\Omega} \varrho \mathbf{v} \cdot \nabla V \, d\mathbf{x} - \int_{\Gamma} \varrho V \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} \, dS - \frac{d}{dt} \int_{\Omega} \varrho V \, d\mathbf{x}, \end{aligned}$$

$\Rightarrow$ the **energy dissipation balance**:

$$\frac{d}{dt} \int_{\Omega} \underbrace{\frac{\varrho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} + \underbrace{\frac{\phi(J)}{J}}_{\text{stored energy}} \, d\mathbf{x} + \int_{\Omega} \underbrace{\delta'_e(J; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v})}_{\text{dissipation rate}} \, d\mathbf{x} = - \int_{\Omega} \underbrace{\varrho \nabla V \cdot \mathbf{v}}_{\text{power of gravitation}} \, d\mathbf{x}.$$

Testing the momentum equation by $\mathbf{v}$ and using the calculus:

$$\begin{aligned}\int_{\Omega} \nabla \mathbf{p} \cdot \mathbf{v} \, d\mathbf{x} &= \int_{\Omega} \phi'(J) \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \phi'(J) \frac{\dot{J}}{J} \, d\mathbf{x} = \int_{\Omega} \left(\left[\frac{\phi(J)}{J} \right]' + \frac{\phi(J)}{J^2} \right) \dot{J} \, d\mathbf{x} \\ &= \int_{\Omega} \left[\frac{\phi(J)}{J} \right]' \dot{J} + \frac{\phi(J)}{J} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{\phi(J)}{J} \right) + \nabla \left(\frac{\phi(J)}{J} \right) \cdot \mathbf{v} + \frac{\phi(J)}{J} \operatorname{div} \mathbf{v} \, d\mathbf{x} \\ &= \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Omega} \operatorname{div} \left(\frac{\phi(J)}{J} \mathbf{v} \right) \, d\mathbf{x} = \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Gamma} \frac{\phi(J)}{J} \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} \, dS.\end{aligned}$$

and testing $V = G\Delta^{-1}\varrho$ by $\frac{\partial}{\partial t} V$ and using the calculus:

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The **basic stability** (a-priori estimates) of the model:

a certain resistivity for compression needed

to prevent a selfgravitational (nonrelativistic) collapse at finite time.

The gravitational energy to be estimated

$$\int_{\Omega} -\varrho V \, d\mathbf{x} \leq \|\varrho\|_{L^2(\Omega)} \|V\|_{L^2(\Omega)} \leq C \max_{\Omega} \varrho_0^2 \int_{\Omega} \frac{1}{J^2} \, d\mathbf{x}.$$

to be handled by use of the stored energy $\phi(J)/J$ by Hölder inequality if $\phi(\cdot)$ grows to $+\infty$ faster than $1/J$ for $J \rightarrow 0+$.

$\Rightarrow$ A-priori bounds (depending on a time horizon) for the kinetic, stored, and gravitational-field energies, as well as for the total dissipation.

Here the $W^{2,r}$ -regularity of the Poisson equation $\Delta V = G\varrho$ on Ω used, with

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Math: B.DUCOMET & E.FEIREISL & Š.NEČASOVÁ (at al.) 2004, 2010

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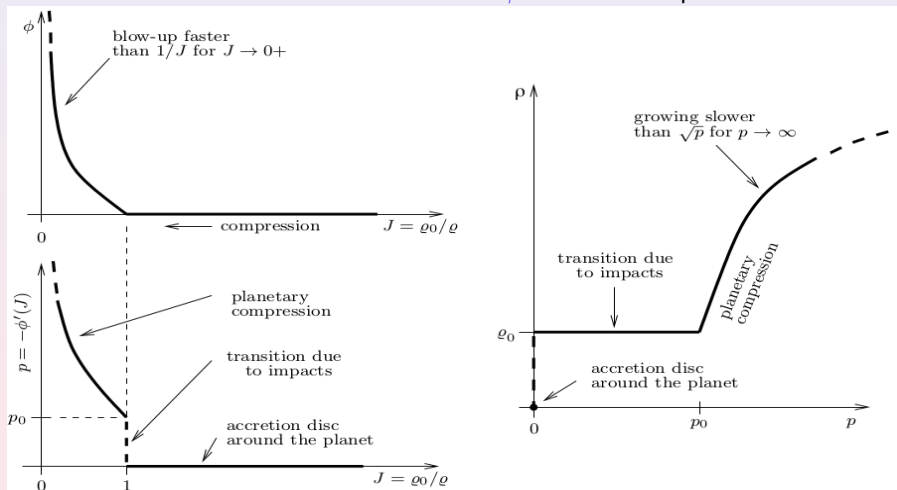
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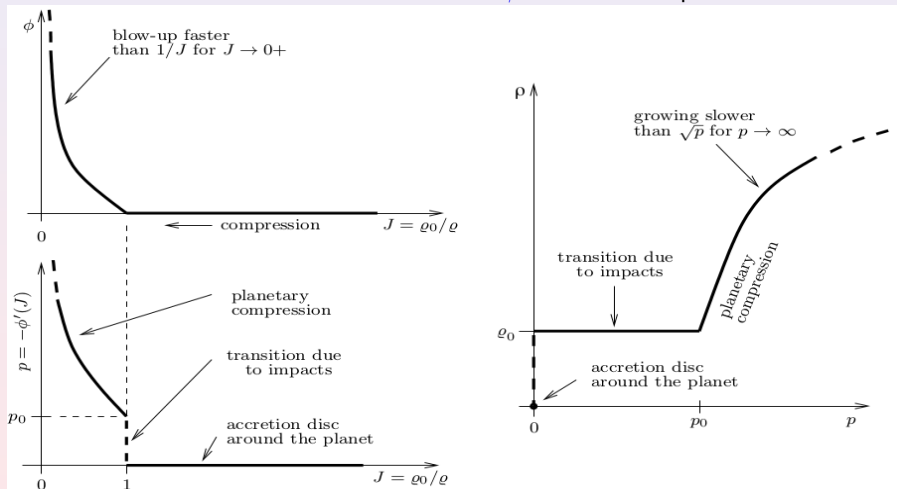
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A schematic scenario of a **convex nonsmooth** ϕ with a blow-up at $J = 0$:



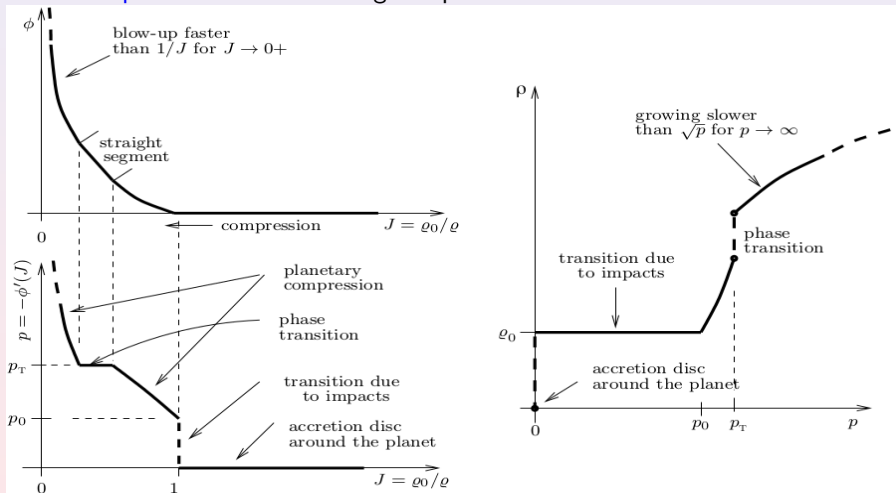
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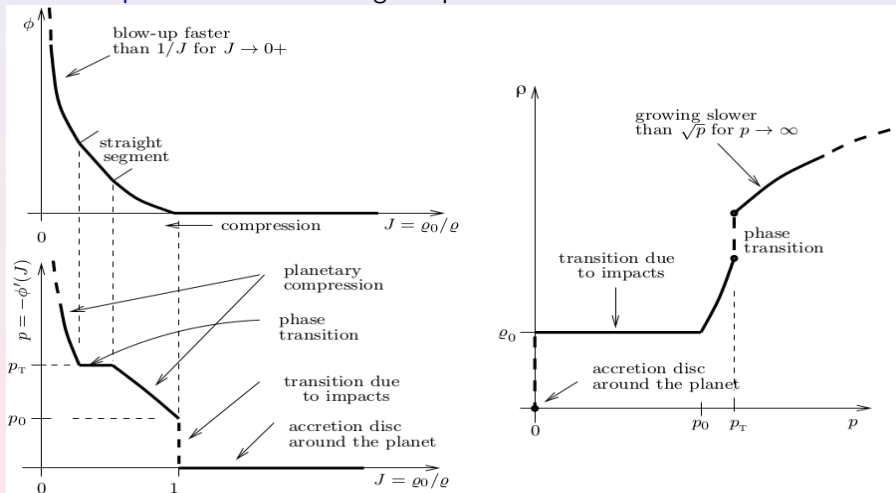
Volumetric phase transitions during compression of silicate rocks:



At the pressure p_T , the rock undergoes a phase transition with **jump of mass density**.

In reality: the main transitions at $p_T = 14$ GPa (in Earth at the depth 440 km) and $p_T = 24$ GPa (in Earth at the depth 670 km).

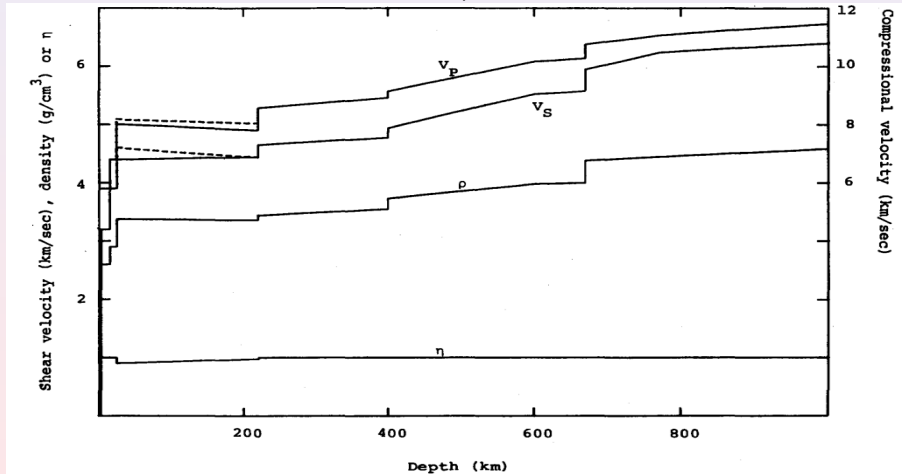
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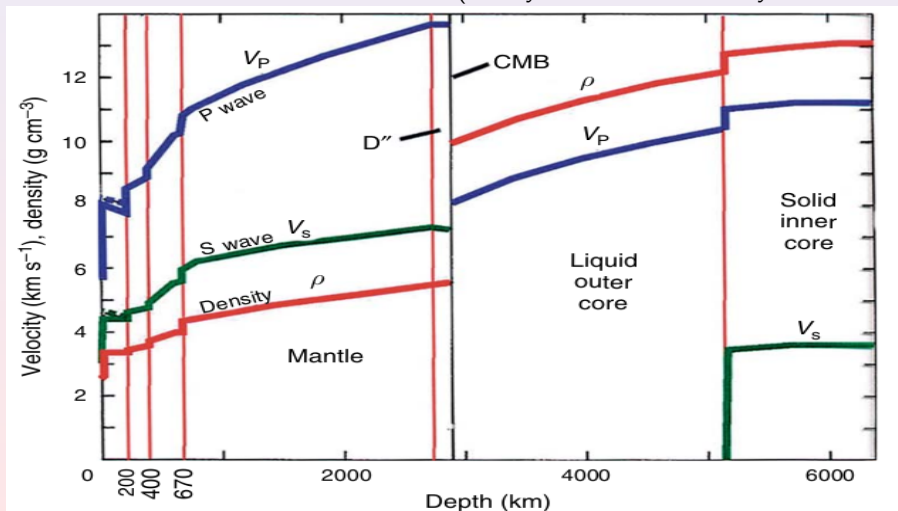
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Another phase transition also in the metallic core (at the depth 5.160 km).

Thermodynamics

Thermodynamics for the 1st fluidic variant, i.e. a referential free energy

$\psi = \psi(J, \theta)$ in $\text{Pa} = \text{J}/\text{m}^3$:

θ = absolute temperature,

pressure $p = -\psi'_J(J, \theta)$,

the referential **entropy** as $-\psi'_\theta(J, \theta)$ in physical units Pa/K

the actual entropy $\eta = -\psi'_\theta(J, \theta)/J$,

the **entropy equation**

$$\frac{\partial \eta}{\partial t} + \text{div}(\mathbf{v}\eta) = \frac{\xi - \text{div} \mathbf{j} + h}{\theta} \quad \text{with } \mathbf{j} = -\kappa(J, \theta) \nabla \theta \quad (\text{i.e. FOURIER's law})$$

$$\text{and } \xi = \delta'_e(J, \theta; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}),$$

where $\kappa = \kappa(J, \theta)$ is the thermal conductivity coefficient in $\text{W}/(\text{m}^2\text{K})$,

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and h = power of heat sources: radiogenic heat (^{232}Th , ^{238}U , ^{40}K , ^{235}K).

$\Rightarrow$ the **heat-transfer equation**

$$c(J, \theta) \dot{\theta} - \text{div}(\kappa(J, \theta) \nabla \theta) = \xi + \theta \psi''_{\theta\theta}(J, \theta) \text{div } \mathbf{v} \quad \text{with } c(J, \theta) = -\theta \frac{\psi''_{\theta\theta}(J, \theta)}{J},$$

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$\Rightarrow$ the **heat-transfer equation in the “enthalpy” formulation**

$$\frac{\partial w}{\partial t} + \text{div}(\mathbf{v}w - \kappa(J, \theta) \nabla \theta) = \xi + h + \gamma'_J(J, \theta) \text{div } \mathbf{v},$$

where $w = \frac{\gamma(J, \theta) - \theta \gamma'_\theta(J, \theta)}{J}$ with $\gamma(J, \theta) = \psi(J, \theta) - \phi(J)$ with $\phi(J) = \psi(J, 0)$.

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The overall **compressible Navier-Stokes-Fourier-Poisson** system:

$$\begin{aligned}\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) &= 0, & \frac{\partial J}{\partial t} &= (\operatorname{div} \mathbf{v})J - \mathbf{v} \cdot \nabla J. \\ \frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} - \delta'_e(J, \theta; \mathbf{e}(\mathbf{v}))) + \nabla p &= \varrho \mathbf{g} \\ &\text{with } p = -\psi'_J(J, \theta), \quad \mathbf{g} = G \int_{\Omega} \frac{\varrho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \cdot)}{|\tilde{\mathbf{x}} - \cdot|^3} d\tilde{\mathbf{x}}, \\ \frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{w} \mathbf{v} - \kappa(J, \theta) \nabla \theta) &= \delta'_e(J, \theta; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} + h \\ &\text{with } w = w(J, \theta) := \frac{\gamma(J, \theta) - \theta \gamma'_\theta(J, \theta)}{J}.\end{aligned}$$

Now, beside mass and momentum conservation, also the **total-energy conservation**:

$$\frac{d}{dt} \left(\underbrace{\int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} + \underbrace{\frac{\phi(J)}{J} + \varrho V}_{\text{stored + gravitational energy}} + \underbrace{w(J, \theta)}_{\text{heat energy}} + \underbrace{\int_{\mathbb{R}^3} \frac{|\nabla V|^2}{2G} d\mathbf{x}}_{\text{energy of gravitational field}} \right) = 0 \quad \text{if } \mathbf{n} \cdot \nabla \theta = 0 \text{ on } \Gamma.$$

$$p = -\psi'_J(J, \theta) \ \& \ \varrho = \varrho_0/J \quad \Rightarrow \quad \text{the state equation } p = -\psi'_J(\varrho_0/\varrho, \theta).$$

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with $p = -\psi'_J(J, \theta), \quad \mathbf{g} = G \int_{\Omega} \frac{\varrho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \cdot)}{|\tilde{\mathbf{x}} - \cdot|^3} d\tilde{\mathbf{x}},$

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with $w = w(J, \theta) := \frac{\gamma(J, \theta) - \theta \gamma'_\theta(J, \theta)}{J}.$

Now, beside mass and momentum conservation, also the **total-energy conservation**:

$$\frac{d}{dt} \left(\underbrace{\int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} + \underbrace{\frac{\phi(J)}{J} + \varrho V}_{\text{stored + gravitational energy}} + \underbrace{w(J, \theta)}_{\text{heat energy}} d\mathbf{x} + \underbrace{\int_{\mathbb{R}^3} \frac{|\nabla V|^2}{2G} d\mathbf{x}}_{\text{energy of gravitational field}} \right) = 0 \quad \text{if } \mathbf{n} \cdot \nabla \theta = 0 \text{ on } \Gamma.$$

$$p = -\psi'_J(J, \theta) \text{ \& \> } \varrho = \varrho_0/J \quad \Rightarrow \quad \text{the state equation } p = -\psi'_J(\varrho_0/\varrho, \theta).$$

The overall **compressible Navier-Stokes-Fourier-Poisson** system:

$$\begin{aligned}\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) &= 0, & \frac{\partial J}{\partial t} &= (\operatorname{div} \mathbf{v})J - \mathbf{v} \cdot \nabla J. \\ \frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} - \delta'_e(J, \theta; \mathbf{e}(\mathbf{v}))) + \nabla p &= \varrho \mathbf{g} \\ \text{with } p &= -\psi'_J(J, \theta), & \mathbf{g} &= G \int_{\Omega} \frac{\varrho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \cdot)}{|\tilde{\mathbf{x}} - \cdot|^3} d\tilde{\mathbf{x}},\end{aligned}$$

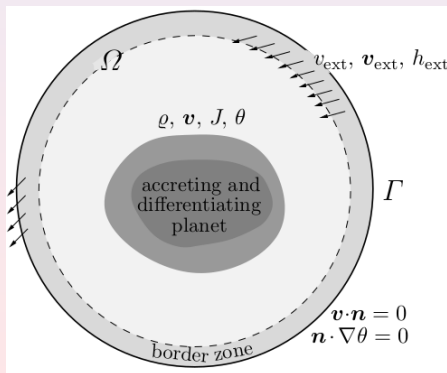
$$\begin{aligned}\frac{\partial w}{\partial t} + \operatorname{div}(w \mathbf{v} - \kappa(J, \theta) \nabla \theta) &= \delta'_e(J, \theta; \mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} + h \\ \text{with } w &= w(J, \theta) := \frac{\gamma(J, \theta) - \theta \gamma'_\theta(J, \theta)}{J}.\end{aligned}$$

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$$p = -\psi'_J(J, \theta) \ \& \ \varrho = \varrho_0/J \quad \Rightarrow \quad \text{the state equation } p = -\psi'_J(\varrho_0/\varrho, \theta).$$

Open thermodynamical system



The incoming (and possibly also outcoming) material in a “border zone”:

$\mathbf{v}_{\text{ext}}$ = velocity of the incoming material,

v_{ext} = volume of the incoming material (with the referential density ϱ_0),

h_{ext} = heat energy (power) of the incoming material,

$$\Rightarrow r_{\text{ext}} = \frac{\varrho_0}{J} v_{\text{ext}} = \text{rate of incoming mass density}.$$

An interpretation of v_{ext} : a **morphogenic growth**.

the standard Kröner-Lee-Liu *multiplicative decomposition* of $\mathbf{F} = \mathbf{F}_e \mathbf{F}_s$
with $\mathbf{F}_e$ the elastic distortion and the inelastic distortion $\mathbf{F}_s = (1+s)^{1/3} \mathbb{I}$
 s = a “volume deviation” due to a morphogenic growth.

$$v_{\text{ext}} = \text{the volume-deviation rate} = \dot{s}/(1+s)$$

$$\text{for a given } v_{\text{ext}} \Rightarrow \text{the volume-deviation ODE: } \dot{s} = v_{\text{ext}}(1+s)$$

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The extended compressible Navier-Stokes-Fourier-Poisson system:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = r_{\text{ext}},$$

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v} - \delta'_e(J, \theta; \mathbf{e}(\mathbf{v}))) + \nabla p = r_{\text{ext}} \mathbf{v}_{\text{ext}} + \varrho \mathbf{g}$$

with $p = -\psi'_J(J, \theta)$, $\mathbf{g} = G \int_{\Omega} \frac{\varrho(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \cdot)}{|\tilde{\mathbf{x}} - \cdot|^3} d\tilde{\mathbf{x}}$,

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$$+ \frac{\gamma(J, \theta)}{J} v_{\text{ext}} + h + h_{\text{ext}} \quad \text{with} \quad w = \frac{\gamma(J, \theta) - \theta \gamma'_\theta(J, \theta)}{J},$$

$$\frac{\partial J}{\partial t} = (\operatorname{div} \mathbf{v} - v_{\text{ext}}) J - \mathbf{v} \cdot \nabla J.$$

The mass and the momentum balances:

$$\underbrace{\frac{d}{dt} \int_{\Omega} \varrho d\mathbf{x}}_{\text{total mass}} = \underbrace{\int_{\Omega} r_{\text{ext}} d\mathbf{x}}_{\text{incoming mass rate}} \quad \text{and} \quad \underbrace{\frac{d}{dt} \int_{\Omega} \varrho \mathbf{v} d\mathbf{x}}_{\text{total momentum}} = \underbrace{\int_{\Omega} r_{\text{ext}} \mathbf{v}_{\text{ext}} d\mathbf{x}}_{\text{incoming momentum rate}}.$$

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The calculus:

$$\begin{aligned}
 \int_{\Omega} \nabla p \cdot \mathbf{v} \, d\mathbf{x} &= \int_{\Omega} \psi'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \int_{\Omega} \phi'(J) (\operatorname{div} \mathbf{v} - \mathbf{v}_{\text{ext}}) + \mathbf{v}_{\text{ext}} \phi'(J) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \int_{\Omega} \phi'(J) \dot{\frac{J}{J}} + \mathbf{v}_{\text{ext}} \phi'(J) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \int_{\Omega} \left(\left[\frac{\phi(J)}{J} \right]' + \frac{\phi(J)}{J^2} \right) \dot{J} + \mathbf{v}_{\text{ext}} \phi'(J) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \int_{\Omega} \left[\frac{\phi(J)}{J} \right]' \dot{J} + \frac{\phi(J)}{J} (\operatorname{div} \mathbf{v} - \mathbf{v}_{\text{ext}}) + \mathbf{v}_{\text{ext}} \phi'(J) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{\phi(J)}{J} \right) + \nabla \left(\frac{\phi(J)}{J} \right) \cdot \mathbf{v} + \frac{\phi(J)}{J} (\operatorname{div} \mathbf{v} - \mathbf{v}_{\text{ext}}) + \mathbf{v}_{\text{ext}} \phi'(J) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Omega} \operatorname{div} \left(\frac{\phi(J)}{J} \mathbf{v} \right) + \mathbf{v}_{\text{ext}} \left(\phi'(J) - \frac{\phi(J)}{J} \right) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} \\
 &= \frac{d}{dt} \int_{\Omega} \frac{\phi(J)}{J} \, d\mathbf{x} + \int_{\Gamma} \frac{\phi(J)}{J} \underbrace{\mathbf{v} \cdot \mathbf{n}}_{=0} \, dS + \int_{\Omega} \mathbf{v}_{\text{ext}} \left(\phi'(J) - \frac{\phi(J)}{J} \right) + \gamma'_J(J, \theta) \operatorname{div} \mathbf{v} \, d\mathbf{x} .
 \end{aligned}$$

The **total-energy balance** for the extended system:

$$\begin{aligned}
 & \frac{d}{dt} \left(\int_{\Omega} \underbrace{\frac{\rho}{2} |\mathbf{v}|^2}_{\text{kinetic energy}} + \underbrace{\frac{\phi(J)}{J}}_{\text{stored energy}} + \underbrace{w(J, \theta)}_{\text{heat energy}} + \underbrace{\rho V}_{\text{gravitational energy}} d\mathbf{x} + \int_{\mathbb{R}^3} \underbrace{\frac{|\nabla V|^2}{2G}}_{\text{energy of gravitational field}} d\mathbf{x} \right) \\
 &= \int_{\Omega} \left(\underbrace{\frac{r_{\text{ext}}}{2} |\mathbf{v}_{\text{ext}}|^2 + v_{\text{ext}} \left(\frac{\psi(J, \theta)}{J} - \psi'_J(J, \theta) \right)}_{\text{rate of kinetic and free and pressure energy incoming from the outer space}} \right. \\
 &\quad \left. + \underbrace{\frac{r_{\text{ext}}}{2} V}_{\text{rate of energy of mass in gravitational field}} + \underbrace{h_{\text{ext}}}_{\text{power of heat coming from outer space}} - \underbrace{\frac{r_{\text{ext}}}{2} |\mathbf{v}_{\text{ext}} - \mathbf{v}|^2}_{\text{power of friction of incoming material}} \right) d\mathbf{x}.
 \end{aligned}$$

The **basic stability** of the model:

allow for $v_{\text{ext}} = v_{\text{ext}}(t, \mathbf{x}, J)$: the most critical is the term

$$v_{\text{ext}} \left(\frac{\psi(J, \theta)}{J} - \psi'_J(J, \theta) \right) = v_{\text{ext}} \left(\frac{\phi(J)}{J} + w(J, \theta) + \theta \frac{\psi'_\theta(J, \theta)}{J} - \psi'_J(J, \theta) \right)$$

which is to be dominated by $\phi(J)/J + w(J, \theta)$ or by $\phi(J)/J + w(J, \theta) \ln w(J, \theta)$

An example:

$$\psi(J, \theta) = \max \left(0, \frac{1}{J^\alpha} - 1 \right) + c_0 \theta (1 - \ln \theta) \quad \text{with some } \alpha > 1.$$

Then the pressure $-\psi'_J(J, \theta) \sim J^{1-\alpha}$ is dominated by $\phi(J)/J$
(the energy-domination of the Kirchhoff stress, J.M. BALL, 1984).

and $\theta \psi'_\theta(J, \theta)/J = c_0 \theta \ln \theta / J$ multiplied by $v_{\text{ext}}(J)$ can be dominated by $w \ln w$
(here $w = c_0 \theta / J$ and $v_{\text{ext}}(J) \sim 1/(1 + \max(0, \ln J))$, needed).

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Multicomponent model

(metals vs silicates)

A simple **mixture model** (without mass-fraction or volume-fraction constraints):

ϱ_i = mass density of the i -component,

$\psi_i = \psi_i(J, \theta)$ a referential free energy of the i -component,

δ_i the viscosity dissipation potential of the i -component,

θ_i = absolute temperature of the i -component,

$\mathbf{v}_i$ velocity of the i -component,

J_i Jacobian of the i -component,

κ_i heat-conductivity coefficient of the i -component,

$f_{ij} = f_{ij}(\varrho_i, \varrho_j)$ viscous-friction coefficient between i - and j -component,

$k_{ij} = k_{ij}(\varrho_i, \varrho_j)$ heat-transfer coefficient between i - and j -component,

$\phi_{ij} = \phi_{ij}(J_i, J_j)$ the “mixing” energy between i - and j -component

to **prevent co-existence of several compressed components in one spot**, e.g.:

$$\phi_{ij}(J_i, J_j) = \kappa \begin{cases} (J_i J_j)^{-\alpha} + \alpha J_i J_j - \alpha - 1 & \text{if } J_i J_j < 1, \\ 0 & \text{if } J_i J_j \geq 1 \end{cases}$$

with some (presumably big) κ (in physical unit Pa=J/m³) and some $\alpha > 0$.

$\mathbf{v}_{\text{ext},i}$, $\mathbf{v}_{\text{ext},i}$, $h_{\text{ext},i}$, $r_{\text{ext},i}$ the incoming i -component

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$\mathbf{v}_{\text{ext},i}$, $\mathbf{v}_{\text{ext},i}$, $h_{\text{ext},i}$, $r_{\text{ext},i}$ the incoming i -component.

The **multicomponent extended compressible Navier-Stokes-Fourier-Poisson system**:

$$\frac{\partial \varrho_i}{\partial t} + \operatorname{div}(\varrho_i \mathbf{v}_i) = r_{\text{ext},i}, \quad \frac{\partial J_i}{\partial t} = (\operatorname{div} \mathbf{v}_i - v_{\text{ext},i}) J_i - \mathbf{v}_i \cdot \nabla J_i$$

$$\frac{\partial}{\partial t}(\varrho_i \mathbf{v}_i) + \operatorname{div}(\varrho_i \mathbf{v}_i \otimes \mathbf{v}_i - [\delta_i]_e'(J_i, \theta_i; \mathbf{e}(\mathbf{v}_i))) + \nabla p_i$$

$$= r_{\text{ext},i} \mathbf{v}_{\text{ext},i} + \varrho_i \mathbf{g} + \sum_{j=1}^n f_{ij}(\mathbf{v}_i - \mathbf{v}_j)$$

$$\text{with } p_i = -[\psi_i]_J'(J_i, \theta_i) - [\phi_{ij}]_{J_i}(J_i, J_j), \quad \mathbf{g} = G \int_{\Omega} \frac{\tilde{\mathbf{x}} - \cdot}{|\tilde{\mathbf{x}} - \cdot|^3} \sum_{i=1}^n \varrho_i(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}},$$

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$$+ \gamma_i(J_i, \theta_i) (\operatorname{div} \mathbf{v}_i - v_{\text{ext},i}) + \frac{\gamma_i(J_i, \theta_i)}{J_i} v_{\text{ext},i} + h_i + h_{\text{ext},i}$$

$$+ \sum_{j=1}^n \left(k_{ij}(\theta_j - \theta_i) + \frac{f_{ij}}{2} |\mathbf{v}_i - \mathbf{v}_j|^2 \right) \quad \text{with} \quad w_i = \frac{\gamma_i(J_i, \theta_i) - \theta_i [\gamma_i]_{\theta}'(J_i, \theta_i)}{J_i}.$$

for $i = 1, \dots, n$.

Mass conservation: $\frac{d}{dt} \int_{\Omega} \sum_{j=1}^n \varrho_j d\mathbf{x} = \sum_{j=1}^n r_j d\mathbf{x}.$

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$$\frac{\partial}{\partial t}(\varrho_i \mathbf{v}_i) + \operatorname{div}(\varrho_i \mathbf{v}_i \otimes \mathbf{v}_i - [\delta_i]_e'(J_i, \theta_i; \mathbf{e}(\mathbf{v}_i))) + \nabla p_i$$

$$= r_{\text{ext},i} \mathbf{v}_{\text{ext},i} + \varrho_i \mathbf{g} + \sum_{j=1}^n f_{ij}(\mathbf{v}_i - \mathbf{v}_j)$$

$$\text{with } p_i = -[\psi_i]_J'(J_i, \theta_i) - [\phi_{ij}]_{J_i}(J_i, J_j), \quad \mathbf{g} = G \int_{\Omega} \frac{\tilde{\mathbf{x}} - \cdot}{|\tilde{\mathbf{x}} - \cdot|^3} \sum_{i=1}^n \varrho_i(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}},$$

$$\frac{\partial w_i}{\partial t} + \operatorname{div}(w_i \mathbf{v}_i - \kappa_i(J_i, \theta_i) \nabla \theta_i) = [\delta_i]_e'(J_i, \theta_i; \mathbf{e}(\mathbf{v}_i)) : \mathbf{e}(\mathbf{v}_i)$$

$$+ \gamma_i(J_i, \theta_i) (\operatorname{div} \mathbf{v}_i - v_{\text{ext},i}) + \frac{\gamma_i(J_i, \theta_i)}{J_i} v_{\text{ext},i} + h_i + h_{\text{ext},i}$$

$$+ \sum_{j=1}^n \left(k_{ij}(\theta_j - \theta_i) + \frac{f_{ij}}{2} |\mathbf{v}_i - \mathbf{v}_j|^2 \right) \quad \text{with} \quad w_i = \frac{\gamma_i(J_i, \theta_i) - \theta_i [\gamma_i]_{\theta}'(J_i, \theta_i)}{J_i}.$$

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Momentum conservation if $f_{ij} = f_{ji}$:

$$\underbrace{\frac{d}{dt} \int_{\Omega} \sum_{j=1}^n \varrho_i \mathbf{v}_i d\mathbf{x}}_{\text{total momentum}} = \underbrace{\int_{\Omega} \sum_{j=1}^n r_{\text{ext},i} \mathbf{v}_{\text{ext},i} d\mathbf{x}}_{\text{incoming momentum rate}} + \underbrace{\int_{\Gamma} \sum_{j=1}^n \phi_{ij}(J_i, J_j) \mathbf{n} dS}_{\text{incoming mixing energy rate}}.$$

The total-energy balance:

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} \sum_{i=1}^n \left(\underbrace{\frac{\varrho_i}{2} |\mathbf{v}_i|^2 + \frac{\phi_i(J_i)}{J_i} + w_i(J_i, \theta_i) + \varrho_i V}_{\text{kinetic, internal, and gravitational energy of particular components}} + \sum_{j=1}^n \underbrace{\phi_{ij}(J_i, J_j)}_{\text{mixing energy}} \right) d\mathbf{x} \right. \\ \left. + \frac{d}{dt} \int_{\mathbb{R}^3} \underbrace{\frac{|\nabla V|^2}{2G}}_{\text{energy of gravitational field}} d\mathbf{x} \right) = \sum_{i=1}^n \int_{\Omega} \left(\underbrace{r_i V}_{\text{rate of energy of mass in gravitational field}} + \underbrace{h_{\text{ext},i}}_{\text{power of heat coming from outer space}} \right. \\ \left. + \underbrace{\frac{r_{\text{ext},i}}{2} |\mathbf{v}_{\text{ext},i}|^2 + \mathbf{v}_{\text{ext},i} \left(\frac{\psi_i(J_i, \theta_i)}{J_i} + [\psi_i]'_J(J_i, \theta_i) \right)}_{\text{rate of kinetic and internal energy coming from the outer space}} - \underbrace{\frac{r_{\text{ext},i}}{2} |\mathbf{v}_{\text{ext},i} - \mathbf{v}_i|^2}_{\text{power of friction of incoming components}} \right) d\mathbf{x}. \end{aligned}$$

Momentum conservation if $f_{ij} = f_{ji}$:

$$\underbrace{\frac{d}{dt} \int_{\Omega} \sum_{j=1}^n \varrho_i \mathbf{v}_i d\mathbf{x}}_{\text{total momentum}} = \underbrace{\int_{\Omega} \sum_{j=1}^n r_{\text{ext},i} \mathbf{v}_{\text{ext},i} d\mathbf{x}}_{\text{incoming momentum rate}} + \underbrace{\int_{\Gamma} \sum_{j=1}^n \phi_{ij}(J_i, J_j) \mathbf{n} dS}_{\text{incoming mixing energy rate}}.$$

The total-energy balance:

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} \sum_{i=1}^n \left(\underbrace{\frac{\varrho_i}{2} |\mathbf{v}_i|^2 + \frac{\phi_i(J_i)}{J_i} + w_i(J_i, \theta_i) + \varrho_i V}_{\text{kinetic, internal, and gravitational energy of particular components}} + \sum_{j=1}^n \underbrace{\phi_{ij}(J_i, J_j)}_{\text{mixing energy}} \right) d\mathbf{x} \right. \\ \left. + \frac{d}{dt} \int_{\mathbb{R}^3} \underbrace{\frac{|\nabla V|^2}{2G}}_{\text{energy of gravitational field}} d\mathbf{x} \right) = \sum_{i=1}^n \int_{\Omega} \left(\underbrace{r_i V}_{\text{rate of energy of mass in gravitational field}} + \underbrace{h_{\text{ext},i}}_{\text{power of heat coming from outer space}} \right. \\ \left. + \underbrace{\frac{r_{\text{ext},i}}{2} |\mathbf{v}_{\text{ext},i}|^2 + \mathbf{v}_{\text{ext},i} \left(\frac{\psi_i(J_i, \theta_i)}{J_i} + [\psi_i]'_J(J_i, \theta_i) \right)}_{\text{rate of kinetic and internal energy coming from the outer space}} - \underbrace{\frac{r_{\text{ext},i}}{2} |\mathbf{v}_{\text{ext},i} - \mathbf{v}_i|^2}_{\text{power of friction of incoming components}} \right) d\mathbf{x}. \end{aligned}$$

The **entropy imbalance**, $\eta_i = -[\psi_i]'_{\theta}(J, \theta_i)$:

$$\text{the algebra } k_{ij} \frac{\theta_j - \theta_i}{\theta_i} + k_{ji} \frac{\theta_i - \theta_j}{\theta_j} = k_{ij} \frac{|\theta_i - \theta_j|^2}{\theta_i \theta_j} \text{ if } k_{ij} = k_{ji} \quad \Rightarrow$$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \sum_{j=1}^n \eta_j d\mathbf{x} &= \underbrace{\int_{\Omega} \sum_{j=1}^n \eta_j d\mathbf{x}}_{\text{total entropy}} \\ &= \int_{\Omega} \sum_{i=1}^n \left(\frac{\kappa_i |\nabla \theta_i|^2}{\theta_i^2} + \frac{[\delta_i]'_{\mathbf{e}}(J_i, \theta_i; \mathbf{e}(\mathbf{v}_i)) : \mathbf{e}(\mathbf{v}_i) + h_i + h_{\text{ext},i}}{\theta_i} + v_{\text{ext},i} \eta_i \right) \\ &\quad + \sum_{i,j=1}^n \left(k_{ij} \frac{|\theta_i - \theta_j|^2}{\theta_i \theta_j} + \frac{f_{ij}}{2\theta_i} |\mathbf{v}_i - \mathbf{v}_j|^2 \right) d\mathbf{x} = \text{entropy production rate} \geq 0. \end{aligned}$$

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Thanks a lot for your attention.