

**FACULTY
OF MATHEMATICS
AND PHYSICS**
Charles University

BACHELOR THESIS

Barbora Piláriková

**Modeling of fault initiation in the ice
shell of Enceladus**

Mathematical Institute of Charles University

Supervisor of the bachelor thesis: doc. RNDr. Ondřej Souček, Ph.D.

Study programme: Mathematics

Study branch: Mathematical modeling

Prague 2024

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature, and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In date

Author's signature

I want to dedicate this thesis to everyone who has been a part of this roller-coaster journey with me.

The biggest thank you goes to Maminka and Tatinko for their unconditional love and support and for making me believe I could be anything I wanted in life. Cheers to my beloved high-school sweethearts who have been making me a better person and a friend since I can remember and convincing me that Matfyz is the coolest thing in the world. A big thank you is addressed to my family who will always be my No.1 inspiration to work harder and smarter. And of course, thank you to the family, I found here in Prague, who have seen me at my best, my worst, and even worse and are always there to pick me up and make any situation better and funnier in every way possible.

A huge thank you goes to Ben Southworth, PhD for providing us with the data and simulations without which we would not be able to get our beautiful results. I would also like to thank all the lecturers and professors who not only taught me how to sit in online lectures for 8 hours straight but also introduced me deeper into the beauties of mathematics and physics. Last but not least, the most appreciation and gratitude goes to my supervisor doc. RNDr. Ondřej Souček, PhD. for always being patient and understanding of all my dumb questions, motivating me every step of the way, and offering endless cups of coffee that were always very much needed.

Title: Modeling of fault initiation in the ice shell of Enceladus

Author: Barbora Piláriková

Institute: Mathematical Institute of Charles University

Supervisor: doc. RNDr. Ondřej Souček, Ph.D., Mathematical Institute of Charles University

Abstract: The 'tiger stripes' on Saturn's icy moon Enceladus are a topic of growing interest. The formation of these four parallel fractures enhances our understanding of the structural dynamics of Enceladus's surface and the moon's potential to harbor life. This study revolves around the formation of new fractures assuming the existence of the first one. We approximate the ice crust by a plate and use the Kirchhoff plate theory simplified to two dimensions, to obtain a fourth-order ordinary differential equation modeling the deformation of the plate. The solution of this equation is obtained by the method of variation of parameters providing us with a function describing the plate's response to different surface load distributions. By investigating the solution profiles for both the approximated point load and the more realistic distributed load and employing the criterion for the maximal bending moment of the plate, we find that the maxima correspond to the positions of the new fracture. Our results indicate that while simple load approximation quite accurately predicts new fracture positions for a reasonable estimate of the elastic shell thickness, the more realistic load model implies a thinner crust more consistent with observations. These findings contribute to a better understanding of the dynamics of Enceladus's surface and spark a thought of how much closer we can get to the real situation.

Keywords: parallel fractures, Kirchhoff plate theory, deflection, bending moment, Enceladus

Contents

Introduction	2
1 Theoretical foundation	4
1.1 Classical Kirchhoff plate theory	4
1.2 Reduction from time-dependent 3D to stationary 2D problem . .	5
1.3 Kirchhoff plate strains	5
1.4 Variational formulation - principle of virtual displacements	7
1.4.1 Virtual strain energy	7
1.4.2 Virtual work	7
1.4.3 Equations of mechanical equilibrium	8
1.4.4 Boundary conditions	11
1.5 Boundary value problem	12
2 Solution	13
2.1 General solution	13
2.2 Particular solution for the general right-hand side	14
2.2.1 Variation of parameters	14
3 Results	19
3.1 Failure criteria	19
3.1.1 Maximum stress	19
3.1.2 Maximum bending moment	19
3.1.3 Crack position for the point load	20
3.2 Load distribution	20
3.2.1 Point load	20
3.2.2 Realistic distributed load	23
3.2.3 Conclusion	26
4 Discussion and conclusions	27
Bibliography	29
List of Figures	30
List of Tables	31

Introduction

One of the topics of ongoing scientific research is Saturn's icy moon Enceladus. This relatively young, 500km-diameter moon stands out not only because of its potential for a habitable environment but also for its unique cryovolcanic activity in the south polar region. This region consists of four parallel fractures called the 'tiger stripes' that are the source of continuous geyser-like eruptions, ejecting water vapor, ice particles, and organic compounds into space, suggesting the presence of a subsurface ocean. Understanding the formation of these fractures is essential for gaining insight into Enceladus's geological processes.

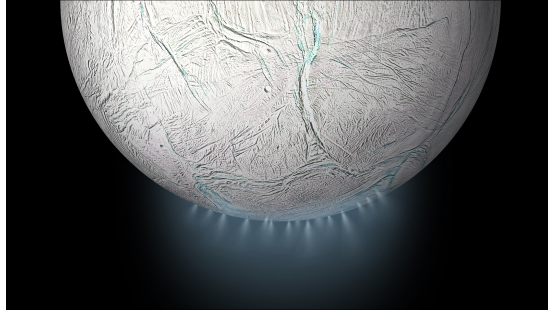


Figure 1: Illustration of Enceladus' plumes in the south polar region from <https://science.nasa.gov/saturn/moons/enceladus/>

Our main interest follows the article Hemingway et al. [2020] where they briefly explain how the first fracture developed and further focus on the position of the new crack from which the next fracture forms. Considering the first tiger stripe was formed due to secular cooling of the ice crust and tidal forces thinning the crust in the south polar region we know that the tensile stresses have been relieved and it became the only active plume ejecting the material from beneath. The ejected material falling back on the moon's surface acts as a surface load bending the part of the ice crust where the material is distributed. There are several ways of interpreting this distribution and we will be looking at two specific ones. Firstly, we rederive the approach from the above-mentioned paper by computing the deflection of ice for a load corresponding to all the ejected material accumulated in one point on the edge. Secondly, we want to approach the situation on the moon more realistically, thus we consider the material distributed over some distance creating a ridge close to the edge of the fracture. The main observation is how the distribution of the load affects the position of the new crack. All this is done in four chapters from the fundamentals of the classical plate theory to the actual comparison of the load's effects.

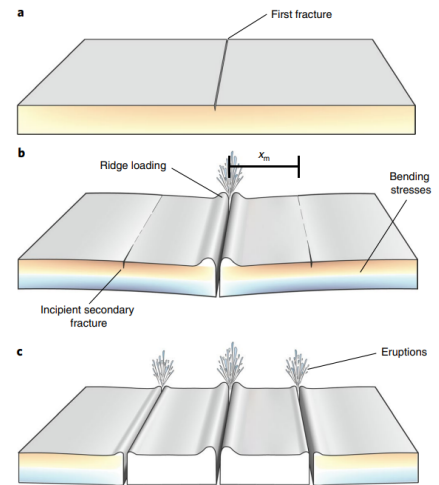


Figure 2: Fracture formation from Hemingway et al. [2020]

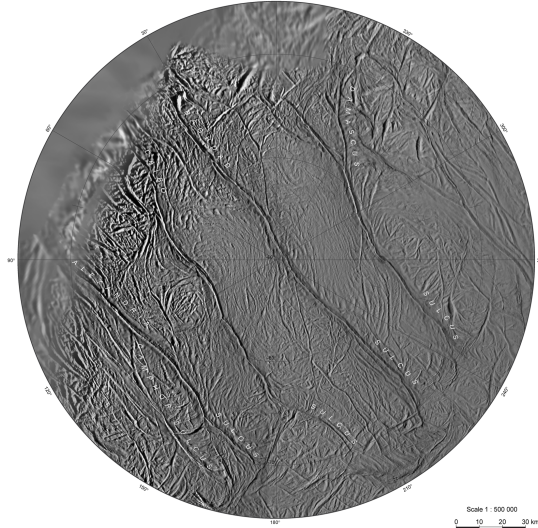


Figure 3: The parallel fractures on the south pole, so-called 'tiger stripes' from <https://science.nasa.gov/saturn/moons/enceladus/>

In these four chapters, the thesis consists of, we cover the essence of Kirchhoff plate theory which is used as an approximation of the ice crust by a semi-infinite plate. The theory defines a plate as a structural element with planar dimensions very large compared to the thickness, thus it can be simplified to two dimensions that are of big use for us. We consider a 2D plate in the xz -plane for which the displacement field and the strains are carefully computed. Further use of these comes up in the variational principle of virtual displacement which is a very strong tool for deriving the leading equation of our problem, the equation of mechanical equilibrium, and the accompanying boundary conditions.

The second chapter purely focuses on solving the fourth-order ODE, derived in the first chapter, by the variation of parameters. This method is chosen because of the need to compute the most general form of the solution possible, as it is further used for different cases of the right-hand side. The classical form of the solution calls for a fundamental system and the homogeneous and particular solution as combinations of the elements of the fundamental system.

Following chapter three, the derived form of the solution comes to use in its full generality. We introduce two special cases of the right-hand side, known as the point and distributed load of the deposition of particles on the moon's surface. By applying the failure criterion consisting of finding the maximum bending moment of the plate, we aim to define the difference between the position of the new crack for the two loads and the reason for that.

After defining the reason for the difference in the positions for different load distributions, which is the inconsistency in the elastic shell thickness, we discuss the effect of this parameter on the position of the new crack in the fourth chapter.

1. Theoretical foundation

In this chapter, we summarize the fundamentals of the classical Kirchhoff plate theory, following closely the exposition from Reddy [2006].

1.1 Classical Kirchhoff plate theory

The Kirchhoff theory is a classical plate theory that focuses on the deformation of elastic plates. As it follows up the Euler-Bernoulli beam theory, only plates with negligible thickness compared to the planar dimensions, are considered. The fundamental assumptions of this theory are the following:

1. The transverse normals to the plate mid-surface remain straight after deformation.
2. The transverse normals are inextensible.
3. The transverse normals remain perpendicular to the middle surface after deformation.

The assumptions 1-3 have implications for the kinematics of Kirchhoff plates, which we now discuss in more detail, which yield a specific form of strain tensor in terms of displacement of the points on the mid-plane.

For a plate in the Cartesian coordinates the xy -plane for $z = 0$ is the mid-plane, h is the thickness of the plate in the z direction, and (u, v, w) denotes the displacement components of a material point originally at coordinates (x, y, z) .

Assumptions 1 and 2 state that the transverse normals do not change along the z -axis. This means that the strain component

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (1.1)$$

must be zero. The third point gives us the condition for the shear strains. It entails that

$$\varepsilon_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0, \quad \varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0. \quad (1.2)$$

As w is independent of z these terms imply a displacement field for time-dependent deformations in the following form

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_0}{\partial x}(x, y, t) \\ v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_0}{\partial y}(x, y, t) \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (1.3)$$

where the terms u_0, v_0 and w_0 represent displacements within the midplane surface.

1.2 Reduction from time-dependent 3D to stationary 2D problem

In view of our application and the character of tiger stripes as long parallel fractures, we can reduce the problem. Firstly, our approach will be purely time-independent, i.e. we will be interested in the plate deformation for a given (time-independent) load. Secondly, we consider a semi-infinite plate in the x -direction and infinite in the y -direction with a small thickness h in the z -direction. Since our goal is to find the position on the top of the plate where a new fracture occurs, we simplify the problem to the coordinates x and z .

Let us denote the new displacement field $\vec{\chi}$ that corresponds to the deformation with respect to the reference configuration, in the following form

$$\begin{aligned} u(x, z) &= u_0(x) - z \frac{\partial w_0}{\partial x}(x), \\ w(x) &= w_0(x). \end{aligned} \tag{1.4}$$

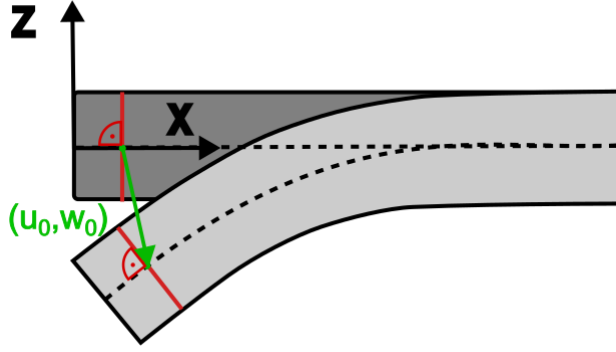


Figure 1.1: Sketch of the problem geometry

1.3 Kirchhoff plate strains

To better understand the system, we must compute the strains (the relative deformation with respect to the reference configuration) associated with the displacement field. We are following the derivation done in Reddy [2006] which starts with finding the displacement gradient $\mathbb{F} = \text{Grad} \vec{\chi} = I + \text{Grad} \vec{U}$ ($\vec{\chi}$ from section 1.2 and $\vec{U} = (u, 0, w)$ is the displacement in the actual configuration) and works towards the non-linear strain tensor. From the previous section where we defined the displacement field, we get the following form of the displacement gradient

$$\mathbb{F} = \begin{pmatrix} \frac{\partial u}{\partial x} + 1 & 0 & \frac{\partial u}{\partial z} \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x} & 0 & 1 \end{pmatrix} \tag{1.5}$$

which gives us the right Cauchy-Green deformation tensor

$$\mathbb{C} = \mathbb{F}^T \mathbb{F} = \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x} + 1 & 0 & \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} & 0 & \left(\frac{\partial u}{\partial z}\right)^2 + 1 \end{pmatrix} \quad (1.6)$$

and using $\mathbb{E} = \frac{1}{2} (\mathbb{C} - \mathbb{I})$ we get our desired strain tensor

$$\mathbb{E} = \frac{1}{2} \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x} & 0 & \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} & 0 & \left(\frac{\partial u}{\partial z}\right)^2 \end{pmatrix}. \quad (1.7)$$

As we can observe $\mathbb{E}$ is a symmetrical matrix and because of the reduction of dimensions, we have only three independent components. Our situation corresponds to small deformations of order ϵ , i.e. $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial z} = \mathcal{O}(\epsilon)$, but moderate rotations. Thus the components representing the deformations of order ϵ^2

$$\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial z}\right)^2, \frac{\partial u}{\partial x} \frac{\partial u}{\partial z}$$

can be omitted due to the terms being negligible compared to the rest of the displacement gradient. But due to the moderate rotations, the component

$$\left(\frac{\partial w}{\partial x}\right)^2$$

cannot be neglected compared to ϵ . We introduce the small-strain-moderate-rotation tensor

$$\varepsilon^K = \begin{pmatrix} \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u}{\partial x} & 0 & \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ 0 & 0 & 0 \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & 0 & 0 \end{pmatrix} \stackrel{1,2}{=} \begin{pmatrix} \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.8)$$

Aiming for the strains to be written in terms of displacement components we substitute in for u and w from Equation 1.4 and get the following expression

$$\varepsilon_{xx}^K = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x}\right)^2. \quad (1.9)$$

The sole remaining strain component is ε_{xx}^K , which can be expressed as the sum of two terms

$$\varepsilon_{xx}^K = \varepsilon_{xx}^{K,0} + z \varepsilon_{xx}^{K,1}. \quad (1.10)$$

Here, $\varepsilon_{xx}^{K,0}$ denotes the so-called *membrane strain*, while $\varepsilon_{xx}^{K,1}$ represents the *bending strain* characterized by

$$\varepsilon_{xx}^{K,0} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x}\right)^2, \quad \varepsilon_{xx}^{K,1} = -\frac{\partial^2 w_0}{\partial x^2}. \quad (1.11)$$

1.4 Variational formulation - principle of virtual displacements

Our main goal for this part is to derive the equations of motion for the bending of the plate and get the expression for specifying the boundary conditions. We will do so by employing the variational principle of virtual displacements in the simplified form

$$0 = \delta U + \delta V.$$

The equations consist of variations of two main quantities which are U as the strain energy and V as the work done by the body against the external forces.

1.4.1 Virtual strain energy

For this section, we first introduce a quantity known as the *Cauchy stress tensor* representing the forces within the body present during deformation (Martinec [2019]). We denote it σ .

The virtual strain energy variation δU for our semi-infinite plate is given by

$$\delta U = \int_0^\infty \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx} \delta \varepsilon_{xx}^K) dz dx = \int_0^\infty \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_{xx} (\delta \varepsilon_{xx}^{K,0} + z \delta \varepsilon_{xx}^{K,1})] dz dx$$

where we first integrate through the thickness of the plate (in the z -direction) and second through the whole midplane (the positive x -axis). Next, introducing the stress resultants; force N_{xx} and moment induced by this force M_{xx}

$$\begin{aligned} N_{xx}(x) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz, \\ M_{xx}(x) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz \end{aligned} \tag{1.12}$$

for easier further computations.

The term for δU can be rewritten in terms of the virtual displacement components using the expressions for the strains from definitions in 1.11 and their variations:

$$\delta \varepsilon_{xx}^{K,0} = \frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x}, \tag{1.13}$$

$$\delta \varepsilon_{xx}^{K,1} = -\frac{\partial^2 \delta w_0}{\partial x^2}. \tag{1.14}$$

Plugging in those variations and the definitions of the stress resultants above 1.12, yields

$$\delta U = \int_0^\infty N_{xx} \left(\frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} \right) + M_{xx} \left(-\frac{\partial^2 \delta w_0}{\partial x^2} \right) dx. \tag{1.15}$$

1.4.2 Virtual work

For the virtual work V , we must consider several forces acting on the plate: the inplane normal stress $\hat{\sigma}_{nn}$ and the transverse shear stress $\hat{\sigma}_{nz}$ that both only act

on the edge (the boundary $x = 0$), the transverse load on the top $q(x)$ and the transverse reaction force of an elastic foundation F_s (Reddy [2006]). All these force components give us the following integral

$$\delta V = - \int_0^\infty (q(x) - F_s) \delta w dx - \int_{-\frac{h}{2}}^{\frac{h}{2}} (\hat{\sigma}_{nn} \delta u_n(0, z) + \hat{\sigma}_{nz} \delta w(0, z)) dz. \quad (1.16)$$

Recalling the kinematic assumption 1.4 we get

$$\begin{aligned} \delta u_n(0, z) &= \delta u_{0n}(0) - z \frac{\partial \delta w_0}{\partial n}(0) \\ \delta w(0, z) &= \delta w_0(0). \end{aligned} \quad (1.17)$$

where the index n signifies the boundary portion of the plate and refers to the outer unit normal to the surface, i.e. $(-1, 0)$ in our case.

From the following form of the equation

$$\delta V = - \int_0^\infty (q(x) - k w_0) \delta w_0 dx - \int_{-\frac{h}{2}}^{\frac{h}{2}} (\hat{\sigma}_{nn} \delta u_{0n} - z \hat{\sigma}_{nn} \frac{\partial \delta w_0}{\partial n} + \hat{\sigma}_{nz} \delta w_0) dz \quad (1.18)$$

we translate the stresses into the stress resultants

$$\hat{N}_{nn}(x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{\sigma}_{nn} dz \quad (1.19)$$

$$\hat{M}_{nn}(x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{\sigma}_{nn} z dz. \quad (1.20)$$

$$\hat{Q}_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{\sigma}_{nz} dz \quad (1.21)$$

that describe the situation on the boundary as they are directly connected to the accompanying boundary variations of displacement.

Finally, the equation comes down to the following

$$\delta V = - \int_0^\infty (q(x) - k w_0) \delta w_0 dx - \left(\hat{N}_{nn} \delta u_{0n} - \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} + \hat{Q}_n \delta w_0 \right) \Big|_{x=0}. \quad (1.22)$$

1.4.3 Equations of mechanical equilibrium

When we put together equations 1.15 and 1.22 we get the expression for the principle of virtual displacements

$$\begin{aligned} 0 = \delta U + \delta V &= \int_0^\infty N_{xx} \frac{\partial \delta u_0}{\partial x} + N_{xx} \left(\frac{\partial w_0}{\partial x} \right) \left(\frac{\partial \delta w_0}{\partial x} \right) + M_{xx} \left(- \frac{\partial^2 \delta w_0}{\partial x^2} \right) dx \\ &- \int_0^\infty (q(x) - k w_0) \delta w_0 dx - \left(\hat{N}_{nn} \delta u_{0n} - \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} + \hat{Q}_n \delta w_0 \right) \Big|_{x=0} \end{aligned} \quad (1.23)$$

from which we want to determine the equations of mechanical equilibrium for each virtual displacement and the accompanying boundary conditions that arise

from further operations. All we know about the boundary conditions for now, is that the plate needs to have a fixed edge at infinity.

Our aim now is to use the fundamental lemma of the calculus of variations for the test functions chosen as the virtual displacements. We must first rid the virtual displacement terms of the differentiation by integrating by parts as done in Reddy [2006] (page 103).

Integration by parts for the first term in 1.23:

$$\int_0^\infty N_{xx} \frac{\partial \delta u_0}{\partial x} dx = [N_{xx} \delta u_0]_0^\infty - \int_0^\infty \frac{\partial N_{xx}}{\partial x} \delta u_0 dx \quad (1.24)$$

Integration by parts for the second term in 1.23:

$$\int_0^\infty N_{xx} \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} dx = \left[N_{xx} \frac{\partial w_0}{\partial x} \delta w_0 \right]_0^\infty - \int_0^\infty \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) \delta w_0 dx \quad (1.25)$$

Double integration by parts for the third term in 1.23 because of the second partial derivative of variation δw_0 :

$$\begin{aligned} - \int_0^\infty M_{xx} \frac{\partial^2 \delta w_0}{\partial x^2} dx &= - \left[M_{xx} \frac{\partial \delta w_0}{\partial x} \right]_0^\infty + \int_0^\infty \frac{\partial M_{xx}}{\partial x} \frac{\partial \delta w_0}{\partial x} dx \\ &= - \left[M_{xx} \frac{\partial \delta w_0}{\partial x} \right]_0^\infty + \left[\frac{\partial M_{xx}}{\partial x} \delta w_0 \right]_0^\infty - \int_0^\infty \frac{\partial^2 M_{xx}}{\partial x^2} \delta w_0 dx \end{aligned} \quad (1.26)$$

All the boundary terms at infinity vanish and the reason for that is the above-mentioned fixed edge condition at infinity. It specifies the actual displacements, thus the virtual displacements and their derivatives must be zero on that portion of the "boundary":

$$\delta u_0 = 0, \quad \delta w_0 = 0, \quad \frac{\partial \delta w_0}{\partial x} = 0, \quad \text{as } x \rightarrow \infty. \quad (1.27)$$

Next, we need to reconcile the stress resultants 1.19-1.21 within the (x, z) coordinate system. It may be observed that the only coordinate not represented in the (x, z) system is n which is the normal to the surface of the broken part of the plate. Geometrically speaking, for the symmetrical semi-infinite problem, n is in the negative x -direction so it corresponds to the vector $n = (-1, 0)$. That being true, we get the following equalities for the stress resultants

$$\begin{aligned} -\hat{N}_{nn} \delta u_{0n} &= \hat{N}_{xx} \delta u_0 \\ \hat{M}_{nn} \frac{\partial \delta w_0}{\partial n} &= -\hat{M}_{xx} \frac{\partial \delta w_0}{\partial x} \\ -\hat{Q}_n \delta w_0 &= \hat{Q}_x \delta w_0. \end{aligned} \quad (1.28)$$

Replacing the terms in 1.23 with those obtained by integration by parts and the new stress resultants, we obtain the following expression

$$\begin{aligned}
0 &= \delta U + \delta V \\
&= - \int_0^\infty \frac{\partial N_{xx}}{\partial x} \delta u_0 dx + \int_0^\infty \left(-\frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) - \frac{\partial^2 M_{xx}}{\partial x^2} - (q(x) - k w_0) \right) \delta w_0 dx \\
&\quad + \left(- (N_{xx} - \hat{N}_{xx}) \delta u_0 - \left(N_{xx} \frac{\partial w_0}{\partial x} + \frac{\partial M_{xx}}{\partial x} - \hat{Q}_x \right) \delta w_0 + (M_{xx} - \hat{M}_{xx}) \frac{\partial \delta w_0}{\partial x} \right) \Big|_{x=0}
\end{aligned} \tag{1.29}$$

where we grouped up the coefficients of each virtual displacement. The boundary terms in zero will further specify the boundary conditions for our problem.

From the integral terms we obtain the Euler-Lagrange equations of the system by setting the coefficients of δu_0 and δw_0 in $(0, \infty)$ to zero separately

$$\delta u_0 : \quad \frac{\partial N_{xx}}{\partial x} = 0 \tag{1.30}$$

$$\delta w_0 : \quad \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) - k w_0 = -q(x). \tag{1.31}$$

An important observation arises from these equations. The equation 1.30 tells us that N_{xx} is constant, i.e.

$$N_{xx}(x) = N_{xx} = \text{const.}$$

throughout the plate, reducing the second term in 1.31 to

$$N_{xx} \frac{\partial^2 w_0}{\partial x^2}.$$

Introducing the equation of mechanical equilibrium

$$\boxed{\frac{\partial^2 M_{xx}(x)}{\partial x^2} + N_{xx} \frac{\partial^2 w_0(x)}{\partial x^2} - \rho g w_0(x) = -q(x)} \tag{1.32}$$

where the term $k = \rho g$ comes from the buoyancy of the ocean under the ice shell (ρ is the density of water, g is the gravitational acceleration), and $q(x)$ is the load applied on the top of the plate.

However, we will be looking for solutions to this equation in terms of w which is hidden in the definitions of N_{xx} and M_{xx} . The only variable that is connected to the stress σ_{xx} in those definitions and containing w , is the strain 1.9. The stress-strain relation, also known as Hooke's law states the following

$$\sigma_{xx} = \frac{E}{1 - \nu^2} \varepsilon_{xx} \tag{1.33}$$

where E stands for Young's modulus and ν is the Poisson's ratio. First, we plug in the equation 1.9 into Hooke's law

$$\sigma_{xx} = \frac{E}{1 - \nu^2} \left(\frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) \tag{1.34}$$

and then the result into 1.12

$$\begin{aligned}
N_{xx}(x) &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 dz \\
&= \frac{Eh}{1-\nu^2} \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) \\
M_{xx}(x) &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) z dz \\
&= -\frac{Eh^3}{12(1-\nu^2)} \frac{\partial^2 w_0}{\partial x^2}
\end{aligned} \tag{1.35}$$

The form of the stress resultants in terms of w_0 is finally plugged into 1.32 and h is replaced by the physical parameter T_e representing the elastic shell thickness of the plate. We arrive at the form

$$\boxed{D \frac{\partial^4 w_0}{\partial x^4} - \underbrace{\frac{ET_e}{1-\nu^2} \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right)}_{N_{xx}=const.} \frac{\partial^2 w_0}{\partial x^2} + \rho g w_0 = q(x)} \tag{1.36}$$

where a new coefficient D emerges from the computation and is given by

$$D = \frac{ET_e^3}{12(1-\nu^2)}. \tag{1.37}$$

1.4.4 Boundary conditions

From the energy principles used earlier, it can be determined what the boundary conditions for our problem are. As a consequence of 1.29-1.31 we get

$$0 = \left(- (N_{xx} - \hat{N}_{xx}) \delta u_0 - \left(N_{xx} \frac{\partial w_0}{\partial x} + \frac{\partial M_{xx}}{\partial x} - \hat{Q}_x \right) \delta w_0 + (M_{xx} - \hat{M}_{xx}) \frac{\partial \delta w_0}{\partial x} \right) \Big|_{x=0} \tag{1.38}$$

that constitutes three conditions combining the primary $(\delta u_0, \delta w_0, \frac{\partial \delta w_0}{\partial x})$ and the associated secondary variables (coefficients) which can be fulfilled by an arbitrary combination of the following boundary conditions

$$N_{xx} - \hat{N}_{xx} = 0 \quad \text{or} \quad \delta u_0 = 0, \tag{1.39}$$

$$N_{xx} \frac{\partial w_0}{\partial x} + \frac{\partial M_{xx}}{\partial x} - \hat{Q}_x = 0 \quad \text{or} \quad \delta w_0 = 0, \tag{1.40}$$

$$M_{xx} - \hat{M}_{xx} = 0 \quad \text{or} \quad \frac{\partial \delta w_0}{\partial x} = 0. \tag{1.41}$$

We can see that different situations may be described using these conditions. From specified moments and forces acting at the boundary to specified actual displacements that give us the virtual displacements equal to zero.

1.5 Boundary value problem

All that is left to consider is the real situation on our plate. Given that our study revolves around a symmetrical problem, where the plate undergoes the first rupture at point $x = 0$, is slightly bent and load is applied from the top which deforms the plate further, we can work on the interval $x \in [0, \infty)$ and make an even extension later. The goal is to set each of the terms in the expression 1.38 to zero so it satisfies the equation. This can be done either by restricting the displacements or by the equilibrium of forces and moments depending on the situation.

We consider a plate with a free end at $x = 0$ and clamped at ∞ , which corresponds to a problem of the existing fracture where the moments are in equilibrium with the applied surface force.

Firstly, let us assume that the prestress or any force acting within the midplane also denoted as $\hat{N}_{xx}$ is zero on the free edge, and by the equation 1.30 it has to be zero within the whole plate,

$$N_{xx} \equiv 0. \quad (1.42)$$

To satisfy the equation 1.38 we need $N_{xx} = \hat{N}_{xx}$, thus we can get rid of the first term in this equation completely as well as the second term in 1.36 that represents the N_{xx} component.

The boundary conditions arising from the free edge imply that the moments of all forces at the boundary must be in equilibrium, thus $M_{xx} = \hat{M}_{xx}$, and since no forces are acting ($\hat{Q}_x = 0$), then the following applies to the moments: $\hat{M}_{xx,x} = 0$. To get these conditions in terms of w_0 we will again use the definition of M_{xx} from 1.12 and Hooke's law 1.33. We set the resulting terms to zero and get the following

$$\frac{\partial^2 w_0(x)}{\partial x^2} = 0, \quad \frac{\partial^3 w_0(x)}{\partial x^3} = 0 \text{ at } x = 0. \quad (1.43)$$

On the other hand, the horizontally clamped edge of the plate implies restrictions on the virtual displacements while allowing for arbitrary assignment of forces and moments. Consequently, under this condition, we seek the deflection $w_0(x)$ and the slope of this function to approach zero as x tends to infinity. Again we specify the boundary conditions in infinity in terms of w_0 , getting

$$w_0(x) = 0, \quad \frac{\partial w_0(x)}{\partial x} = 0 \text{ as } x \rightarrow \infty. \quad (1.44)$$

As we may notice the only direction in which changes are happening is the x direction thus we denote $w_0(x)$ as $w(x)$ from now on and we can rewrite all the partial derivatives to total differentials and get a linear non-homogeneous fourth-order ordinary differential equation (ODE). The final form of our boundary value problem with a general right-hand side is the following

$$D \frac{d^4 w(x)}{dx^4} + \rho g w(x) = q(x) \quad \text{in } (0, \infty) \quad (1.45)$$

$$\frac{d^2 w(x)}{dx^2} = 0, \quad \frac{d^3 w(x)}{dx^3} = 0 \quad \text{in } 0 \quad (1.46)$$

$$w(x) = 0, \quad \frac{dw(x)}{dx} = 0 \quad \text{in } \infty. \quad (1.47)$$

2. Solution

In the previous chapter, we derived an equation that describes the bending of a plate with respect to the load applied to it. Let us now find the general form of the bending function $w(x)$. The function is further specified using variation of parameters and the boundary conditions to become the solution for the boundary value problem obtained in section 1.5.

2.1 General solution

The objective of this section is to find the solution to the linear nonhomogeneous fourth-order ODE 1.45 on the domain $(0, \infty)$. The general form of the solution for a higher-order ODE with constant coefficients that we are looking for is

$$w(x) = w_h(x) + w_p(x) \quad (2.1)$$

where $w_h(x)$ is the solution of the homogeneous equation and $w_p(x)$ is the particular solution of 1.45.

First, we compute the fundamental system of the homogeneous equation

$$D \frac{d^4 w}{dx^4} + \rho g w = 0 \quad (2.2)$$

by solving the characteristic equation

$$\lambda^4 = \frac{-\rho g}{D}. \quad (2.3)$$

Equation 2.3 has four complex roots

$$\lambda_i = \sqrt[4]{\frac{\rho g}{D}} e^{\frac{\pi i}{4} + \frac{k\pi i}{2}} = \sqrt[4]{\frac{\rho g}{D}} \frac{\pm 1 \pm i}{\sqrt{2}} \text{ for } i = 0, 1, 2, 3$$

that can be rewritten into the real form and construct the following fundamental system

$$FS = \left\{ \underbrace{e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right)}_{w_1(x)}, \underbrace{e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right)}_{w_2(x)}, \underbrace{e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right)}_{w_3(x)}, \underbrace{e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right)}_{w_4(x)} \right\} \quad (2.4)$$

where

$$\alpha = \sqrt[4]{\frac{4D}{\rho g}}. \quad (2.5)$$

Homogeneous solution $w_h(x)$ is a linear combination of the terms in the fundamental system with a_i as real constants

$$\begin{aligned} w_h(x) = & a_1 e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + a_2 e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + a_3 e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + a_4 e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right). \end{aligned} \quad (2.6)$$

Furthermore, we construct the particular solution $w_p(x)$. It is again a combination of the terms in the fundamental system $w_i(x)$ but this time the coefficients $C_i(x)$ are functions of the variable x , thus the form

$$w_p(x) = \sum_{i=1}^4 C_i(x) w_i(x) \quad \text{for } i = 1, 2, 3, 4 \quad (2.7)$$

which extends to

$$\begin{aligned} w_p(x) = & C_1(x) e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_2(x) e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + C_3(x) e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_4(x) e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right). \end{aligned} \quad (2.8)$$

The general solution then looks like

$$\begin{aligned} w(x) = & a_1 e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + a_2 e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + a_3 e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + a_4 e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + C_1(x) e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_2(x) e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + C_3(x) e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_4(x) e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right). \end{aligned} \quad (2.9)$$

2.2 Particular solution for the general right-hand side

Since we aim to compute the solution for the general $q(x)$ we need to determine the form of the coefficients $C_i(x)$ in 2.8. This is obtained by using the variation of parameters and boundary conditions.

2.2.1 Variation of parameters

We proceed by variation as done in Černý and Pokorný [2016]. In short, we need to find a four times differentiable solution that satisfies the boundary conditions 1.46 and 1.47. In this process, with every differentiation of $w_p(x)$, we set the term including the first derivatives of the parameters $C'_i(x)$ to zero, e.g. for the first derivative

$$\begin{aligned} 0 = & C'_1(x) e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C'_2(x) e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + C'_3(x) e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C'_4(x) e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right). \end{aligned} \quad (2.10)$$

After substituting for terms $w(x)$ and $w^{(4)}(x)$ into the equation 1.45 we get that all terms either satisfy the homogeneous equation or the method (variation of parameters) sets them equal to zero. Except for the term with the highest derivative of $w_i(x)$. We end up with a system of four algebraic equations for $C'_i(x)$

$$\begin{bmatrix} w_1(x) & w_2(x) & w_3(x) & w_4(x) \\ w_1'(x) & w_2'(x) & w_3'(x) & w_4'(x) \\ w_1''(x) & w_2''(x) & w_3''(x) & w_4''(x) \\ w_1'''(x) & w_2'''(x) & w_3'''(x) & w_4'''(x) \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \\ C_3'(x) \\ C_4'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{q(x)}{D} \end{bmatrix} \quad (2.11)$$

from which we can compute the derivatives of the parameters.

By solving the above system of algebraic equations we get the results given by

$$C_1'(x) = -\frac{\alpha^3}{8D} e^{-\frac{x}{\alpha}} \left(\cos\left(\frac{x}{\alpha}\right) + \sin\left(\frac{x}{\alpha}\right) \right) q(x) \quad (2.12)$$

$$C_2'(x) = \frac{\alpha^3}{8D} e^{-\frac{x}{\alpha}} \left(\cos\left(\frac{x}{\alpha}\right) - \sin\left(\frac{x}{\alpha}\right) \right) q(x) \quad (2.13)$$

$$C_3'(x) = \frac{\alpha^3}{8D} e^{\frac{x}{\alpha}} \left(\cos\left(\frac{x}{\alpha}\right) - \sin\left(\frac{x}{\alpha}\right) \right) q(x) \quad (2.14)$$

$$C_4'(x) = \frac{\alpha^3}{8D} e^{\frac{x}{\alpha}} \left(\cos\left(\frac{x}{\alpha}\right) + \sin\left(\frac{x}{\alpha}\right) \right) q(x). \quad (2.15)$$

The apparent way of computing the terms for $C_i(x)$ is integrating from 0 to x as follows

$$C_i(x) = \int_0^x C_i'(\xi) d\xi + C_i^0 \quad (2.16)$$

where C_i^0 are integration constants that will be specified using the boundary conditions. The first condition in 1.47 tells us that

$$\lim_{x \rightarrow \infty} w_p(x) = 0. \quad (2.17)$$

Two distinct cases need to be considered. The first case is for the parameters $C_i(x)$ where $i = 1, 2$, associated with the growing exponential. The terms in $w_p(x)$ only go to zero if the limit $\lim_{x \rightarrow \infty} C_i(x)w_i = 0$ holds for $i = 1, 2$. This in particular implies that also $\lim_{x \rightarrow \infty} C_i(x) = 0$ must hold for $i = 1, 2$. Consequently, we get the form of the integration constants as

$$C_i^0 = -\int_0^\infty C_i'(\xi) d\xi \quad \text{for } i = 1, 2, \quad (2.18)$$

and substituting back into the integral we get

$$C_i(x) = -\int_x^\infty C_i'(\xi) d\xi \quad \text{for } i = 1, 2 \quad (2.19)$$

where the integration constants C_i^0 specified the limits of integration.

The second case concerns the other two parameters $C_i(x)$ where $i = 3, 4$. The limit 2.17 holds for all functions $C_i(x)$ that tend to infinity slower than $e^{\frac{x}{\alpha}}$, so, in general they need not be bounded. For $i = 3, 4$ we again get the expression 2.16 where the C_i^0 will be specified in a bit.

Another important task is to check if the other boundary condition

$$\lim_{x \rightarrow \infty} \frac{dw_0(x)}{dx} = 0$$

holds for the found expressions for the parameters. We will use the expressions for $w_p'(x)$ computed by the variation of 2.8

$$\begin{aligned}
w_p'(x) = & C_1'(x)w_1(x) + \frac{1}{\alpha}C_1(x)w_1(x) - \frac{1}{\alpha}C_1(x)w_2(x) \\
& + C_2'(x)w_2(x) + \frac{1}{\alpha}C_2(x)w_2(x) + \frac{1}{\alpha}C_2(x)w_1(x) \\
& + C_3'(x)w_3(x) - \frac{1}{\alpha}C_3(x)w_3(x) - \frac{1}{\alpha}C_3(x)w_4(x) \\
& + C_4'(x)w_4(x) - \frac{1}{\alpha}C_4(x)w_4(x) + \frac{1}{\alpha}C_4(x)w_3(x).
\end{aligned} \tag{2.20}$$

We know from the previous paragraphs, that the products $C_i(x)w_i$ as well as $C_i(x)w_j$ for $i, j \in \{1, 2\}$ or $i, j \in \{3, 4\}$ go to zero as x tends to infinity and the products $C_i'(x)w_i(x)$ are set to zero by variation of parameters.

We successfully found the expressions for the parameters $C_1(x)$ and $C_2(x)$ as follows

$$C_1(x) = \frac{\alpha^3}{8D} \int_x^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) + \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \tag{2.21}$$

$$C_2(x) = -\frac{\alpha^3}{8D} \int_x^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \tag{2.22}$$

Let us continue with the second case where the last task is to determine the explicit form of the integration constants denoted C_3^0 and C_4^0 from

$$C_3(x) = \frac{\alpha^3}{8D} \left(\int_0^x e^{\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi + C_3^0 \right) \tag{2.23}$$

$$C_4(x) = \frac{\alpha^3}{8D} \left(\int_0^x e^{\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) + \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi + C_4^0 \right) \tag{2.24}$$

with the help of the conditions 1.46. The computation of the derivatives of $w_p(x)$ from the variation now becomes useful again as we need to compute the second and third derivatives at the point $x = 0$. As we know these expressions are combinations of $w_i''(x)$ and $w_i'''(x)$ respectively with coefficients $C_i(x)$. The values of $w_i''(0)$ and $w_i'''(0)$ are

$$w_1''(0) = 0, \quad w_2''(0) = \frac{2}{\alpha^2}, \quad w_3''(0) = 0, \quad w_4''(0) = -\frac{2}{\alpha^2}, \tag{2.25}$$

$$w_1'''(0) = -\frac{2}{\alpha^3}, \quad w_2'''(0) = \frac{2}{\alpha^3}, \quad w_3'''(0) = \frac{2}{\alpha^3}, \quad w_4'''(0) = \frac{2}{\alpha^3}, \tag{2.26}$$

which simplifies the terms for $w_p''(x)$ and $w_p'''(x)$ to

$$w_p''(0) = \frac{2}{\alpha^2}C_2(x)\Big|_{x=0} - \frac{2}{\alpha^2}C_4(x)\Big|_{x=0} \tag{2.27}$$

$$w_p'''(0) = -\frac{2}{\alpha^3}C_1(x)\Big|_{x=0} + \frac{2}{\alpha^3}C_2(x)\Big|_{x=0} + \frac{2}{\alpha^3}C_3(x)\Big|_{x=0} + \frac{2}{\alpha^3}C_4(x)\Big|_{x=0} \tag{2.28}$$

As the conditions 1.46 say, these terms are set to zero so we can easily figure out the expressions for integration constants. In the first case we get that $C_2(x)\Big|_{x=0} = C_4(x)\Big|_{x=0}$ where after substituting in the zero, we get that the expression for $C_2(x)$ is an integral from 0 to ∞ and for $C_4(x)$ we get an integral from 0 to 0 which integrates to 0 leaving us only with the integration constant. Thus the integration constant C_4^0 attains the form

$$C_4^0 = - \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi. \quad (2.29)$$

We will proceed with the same process for C_3^0 . For $C_1(x)$ we get an integral from 0 to ∞ , $C_2(x)\Big|_{x=0}$ is equal to C_4^0 which we know the expression for, leaving us with $C_3(x)\Big|_{x=0}$ where again the integral from 0 to 0 integrates to 0 with the constant C_3^0 left. By substituting all this into the equation 2.28 and setting it to zero as the second condition in 1.46 we get

$$\begin{aligned} 0 = & - \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) + \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \\ & - \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \\ & + C_3^0 - \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \end{aligned} \quad (2.30)$$

and by expressing the C_3^0 separately we get the final form

$$C_3^0 = \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(3 \cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi. \quad (2.31)$$

Verification of all terms $C_i(x)$ can be done by substituting them into the system of equations 2.11 where we get the corresponding right-hand side for each derivative.

The final form of the particular solution satisfying all four boundary equations is

$$\begin{aligned} w_p(x) = & C_1(x) e^{\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_2(x) e^{\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \\ & + C_3(x) e^{-\frac{x}{\alpha}} \cos\left(\frac{x}{\alpha}\right) + C_4(x) e^{-\frac{x}{\alpha}} \sin\left(\frac{x}{\alpha}\right) \end{aligned} \quad (2.32)$$

with the parameters

$$C_1(x) = \frac{\alpha^3}{8D} \int_x^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) + \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \quad (2.33)$$

$$C_2(x) = -\frac{\alpha^3}{8D} \int_x^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \quad (2.34)$$

$$\begin{aligned} C_3(x) &= \frac{\alpha^3}{8D} \int_0^x e^{\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \\ &\quad + \frac{\alpha^3}{8D} \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(3\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \end{aligned} \quad (2.35)$$

$$\begin{aligned} C_4(x) &= \frac{\alpha^3}{8D} \int_0^x e^{\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) + \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \\ &\quad - \frac{\alpha^3}{8D} \int_0^\infty e^{-\frac{\xi}{\alpha}} \left(\cos\left(\frac{\xi}{\alpha}\right) - \sin\left(\frac{\xi}{\alpha}\right) \right) q(\xi) d\xi \end{aligned} \quad (2.36)$$

Let us note that we need to ensure integrability in the above expression. We will not look for a precise characterization of $q(x)$ but notice that a sufficient condition that gives good mathematical meaning to all the above expressions is the condition that the function $q(x)$ has compact support on the interval $(0, \infty)$.

Last but not least one can verify that the homogeneous solution is trivial for our boundary conditions since no force or momenta are acting on either edge. So the final form of our bending function $w(x)$ satisfying the boundary conditions consists only of the particular solution $w_p(x)$ in the form 2.8, thus $w(x) = w_p(x)$ with the constant functions $C_i(x)$ in 2.33-2.36.

3. Results

The objective of this chapter is to compare methods and results with those in Hemingway et al. [2020]. First, we look at the accuracy of two apparently different failure criteria and confirm they are indeed identical. Second, we focus on how the load distribution affects the position of the new crack from which the fracture develops. We will be using the notation $w(x)$ instead of $w_p(x)$ to denote the bending function in the following chapters.

3.1 Failure criteria

There are two strategies on how to approach our problem. One focuses on finding the point of the maximum stress on the plate's surface where the crack occurs and a new fracture develops. The second one focuses on the maximum bending moment which is slightly unusual from the physics point of view. We aim to compare these two approaches for a special case of the point load and show that they yield the same position of the new crack.

3.1.1 Maximum stress

Starting from the physics point of view it is more natural to look at the maximum stress on the plate's surface and compare it with the critical extensional stress for ice ($\sigma_{crit} = 1\text{MPa}$, Petrenko and Whitworth [1999]).

The expression for the stress function is obtained from the constitutional relation 1.33 by substituting in for the strain component ε_{xx} 1.10 as follows

$$\sigma_{xx}(x, z) = \frac{E}{1 - \nu^2} \left(\frac{\partial u_0(x)}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0(x)}{\partial x} \right)^2 - z \frac{\partial^2 w_0(x)}{\partial x^2} \right) \quad (3.1)$$

and we want to know when the following equation, corresponding to the maximal extensional stress on the surface of the plate, holds

$$\frac{\partial \sigma_{xx}(x, \frac{h}{2})}{\partial x} = 0. \quad (3.2)$$

Recalling the identity for N_{xx} from 1.42 we can observe that the first two terms in equation 3.1 sum up to zero and thus 3.2 in our problem takes the form

$$-\frac{Eh}{2(1 - \nu^2)} w'''(x) = 0. \quad (3.3)$$

3.1.2 Maximum bending moment

In comparison to the first method, the maximum bending moment is easier to solve as done in Hemingway et al. [2020], as it contains fewer terms from the beginning. By using the constitutive relation 1.33 and substituting for the strains into M_{xx} in 1.12 we again end up with an equation only linearly dependent on the bending function

$$M_{xx}(x) = D \frac{d^2 w(x)}{dx^2}. \quad (3.4)$$

For the maximum bending moment, we need to solve

$$\frac{dM_{xx}(x)}{dx} = 0 \quad (3.5)$$

that implies that the third derivative $w'''(x)$ needs to be zero. Thus we are solving an equation

$$Dw'''(x) = 0. \quad (3.6)$$

3.1.3 Crack position for the point load

Comparing 3.3 and 3.6 we get the same condition for the third derivative to be zero, thus we are solving the equation

$$w'''(x_m) = 0, \quad (3.7)$$

where x_m is the new position of the crack.

Taking the expression for $w(x)$ computed in section 2.2 and substituting in for the point load 3.12 we get the expression 3.15

$$w(x) = \frac{V_p \alpha^3}{4D} e^{-\frac{x}{\alpha}} \cos \frac{x}{\alpha}. \quad (3.8)$$

Computing the third derivative of this expression gives us

$$w'''(x_m) = \frac{2}{\alpha^3} e^{-\frac{x_m}{\alpha}} \left(\cos \left(\frac{x_m}{\alpha} \right) - \sin \left(\frac{x_m}{\alpha} \right) \right) \quad (3.9)$$

which can be simplified to

$$1 - \tan \left(\frac{x_m}{\alpha} \right) = 0. \quad (3.10)$$

From this equation, we can easily determine the value of x_m that satisfies it:

$$x_m = \alpha \frac{\pi}{4}. \quad (3.11)$$

3.2 Load distribution

Load distribution on the plate's surface is the way how we get closer to the real situation. Starting from the simplest form of the load as done in Hemingway et al. [2020], getting to the more complicated approximation of the load function out of our curiosity and potential for future studies.

3.2.1 Point load

Starting with the simplest load distribution as in Hemingway et al. [2020] where the authors consider the mass accumulated all in one point on the plate's broken edge, thus the term *point load*. The main goal here is to use our approach of finding the solution of the accompanying equation 1.45 and the same load and physical parameters used in the article and compare the results.

Let us take the bending function 2.8 and the special right-hand side

$$q(x) = \frac{V_p}{2} \delta_0(x), \quad (3.12)$$

where V_p takes on the amount of mass accumulated at the edge of the broken plate and $\frac{1}{2}$ is needed because we are only taking half of the mass fallen onto the surface (symmetrical problem). We take $V_p = 3 \times 10^8 \text{ Nm}^{-1}$ from Hemingway et al. [2020] and use the Dirac delta distribution as it best describes the situation and also satisfies the compact support condition introduced in section 2.2.1.

When plugging in the load 3.12 into the equation 2.8 we have to compute the expressions for $C_i(x)$, $i \in \{1, 2, 3, 4\}$ to get the exact expression for $w(x)$. Upon evaluating the integrals, we find that the Dirac delta distribution forces the terms $C_1(x)$ and $C_2(x)$ to vanish for $x > 0$, as we integrate these terms over the interval from x to ∞ which does not contain zero.

Similarly for $C_4(x)$ where the Dirac delta distribution sets the whole integrand in both terms to the point $x = 0$ making them both equal to 1 and we get

$$C_4(x) = \frac{V_p \alpha^3}{16D} - \frac{V_p \alpha^3}{16D} = 0. \quad (3.13)$$

The only remaining term is $C_3(x)$ where the Dirac delta distribution sets both integrands to the point $x = 0$ again and we are left with

$$C_3(x) = \frac{V_p \alpha^3}{16D} + \frac{3V_p \alpha^3}{16D} = \frac{V_p \alpha^3}{4D}. \quad (3.14)$$

We plug the results for $C_i(x)$ back into $w(x) = \sum_{i=1}^4 C_i(x)w_i(x)$ and the expression for the bending function is then reduced to

$$w(x) = \frac{V_p \alpha^3}{4D} e^{-\frac{x}{\alpha}} \cos \frac{x}{\alpha} \quad (3.15)$$

which corresponds to the results derived in Turcotte and Schubert [2002] for the same type of problem. This equation for the bending function is used for further computation of the position of the new crack.

As concluded in section 3.1.3, the position of the new crack, corresponding to the point of the maximum bending moment (red point in figure 3.1) for point load, is at

$$x_m = \alpha \frac{\pi}{4} = 33\,898.87 \text{ m} \quad (3.16)$$

as may be seen in the following figure along with the profile of the function $w(x)$.

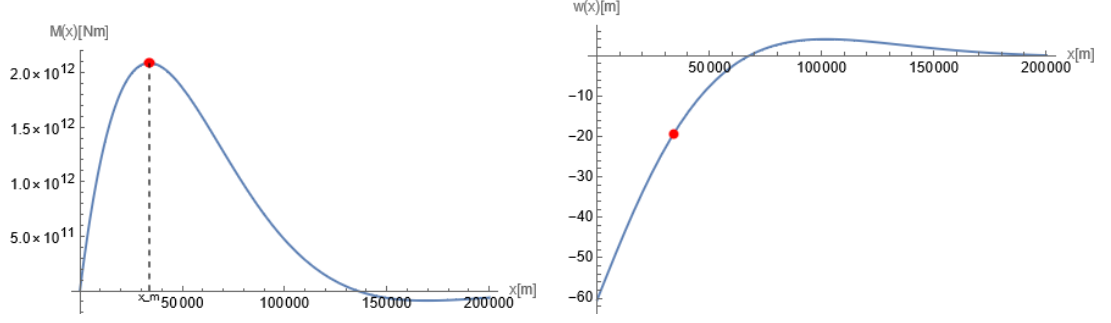


Figure 3.1: Left: The bending moment for point load. Right: The bending function for point load. In both figures, the red point signifies the position of the new crack.

The result corresponds to Hemingway et al. [2020] for T_e as defined below. The computations are done for the physical parameters defined in section 1.4.3 of values in Table 3.1.

Parameter	Symbol	Value
Elastic Thickness	T_e	≈ 5 km
Young's Modulus for ice	E	9 GPa
Poisson's Ratio	ν	0.25
Gravitational Acceleration	g	0.113 ms^{-2}
Water Density	ρ	1020 kgm^{-3}
Flexural Rigidity	D	$\frac{E T_e^3}{12(1 - \nu^2)}$

Table 3.1: Table of Physical Parameters

It is important to notice that in Hemingway et al. [2020] they in fact used $q(x) \equiv 0$ and considered the term

$$\hat{Q}_n = \frac{V_p}{2}$$

instead ($\hat{Q}_n$ from 1.21). This expression is equivalent to the condition

$$Dw'''(0) = \frac{V_p}{2}$$

replacing our $w'''(0) = 0$, see 1.46.

3.2.2 Realistic distributed load

Hemingway et al. [2020] assume, the approximation of the load on the plate's surface by point load gives results that are accurate enough for specific parameters. However, as we consider a real object in space we should consider a more realistic situation.

Our problem is defined as an existing fracture from which particles from the ocean below the ice crust erupt. Around 90% of these particles fall ballistically back on the surface creating a ridge close to the fracture. We use data from Southworth et al. [2019] characterizing a ballistic model of the deposition rate of these particles. This dataset can be parametrized as a 1D function depending on the shortest distance to the crack, see Figure 3.3.

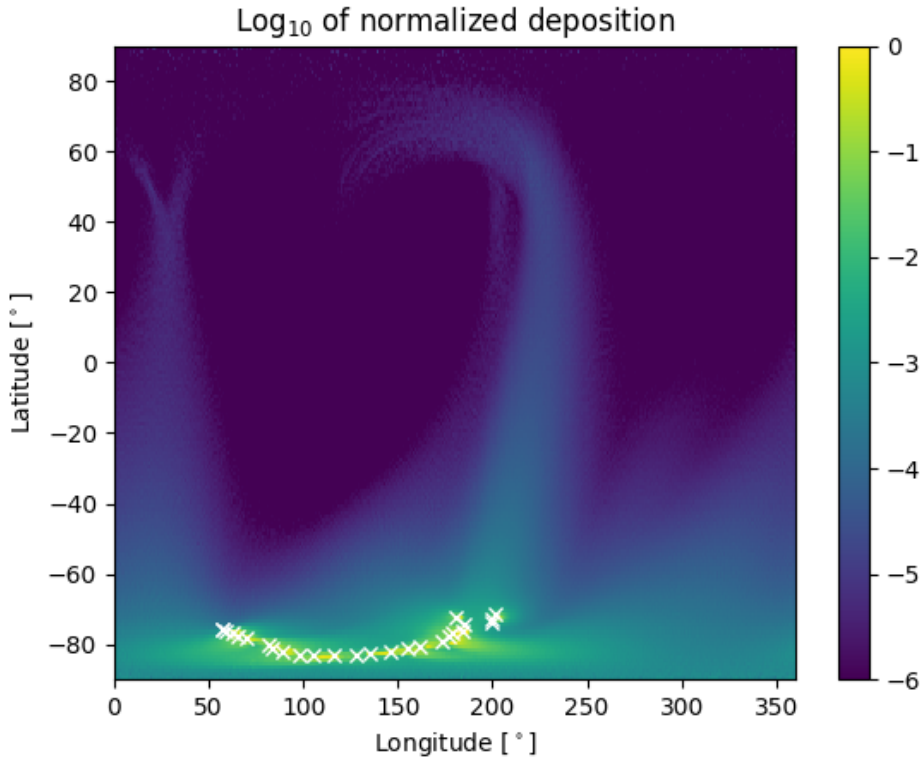


Figure 3.2: Decadic logarithm of normalized surface mass deposition rate as a function of position on the surface of Enceladus for Cairo sulcus as the only active source of plume ejecta (Southworth et al. [2019]) zoomed to the south-polar region.

The 1D parametrization is a normalized power function of a fifth-degree polynomial in the following form

$$q_d(x) = q_0 \cdot 10^{Bx_q + Cx_q^2 + Dx_q^3 + Ex_q^4 + Fx_q^5} \quad (3.17)$$

with constants of values

$$\begin{aligned} B &= -0.45, \quad C = 3.43 \times 10^{-2}, \\ D &= -1.39 \times 10^{-4}, \quad E = 2.64 \times 10^{-5}, \quad F = -1.88 \times 10^{-7}, \end{aligned} \quad (3.18)$$

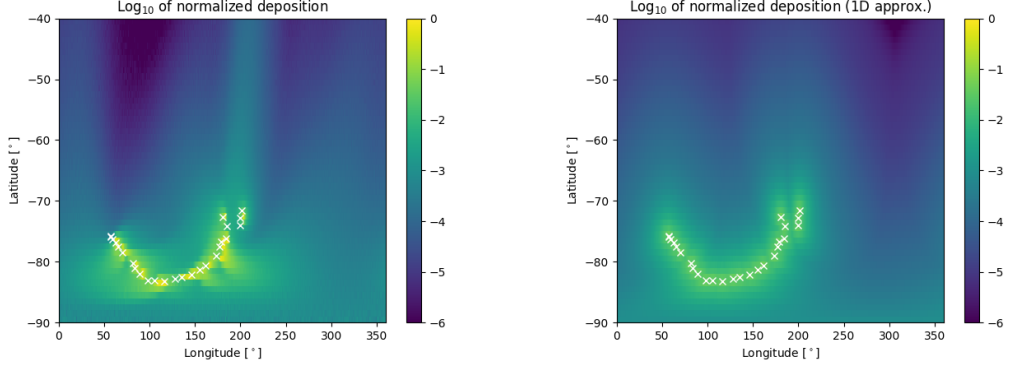


Figure 3.3: Left: Zoomed in 3.2. Right: Parameterization of this dataset by a one-dimensional fit of the deposition function in terms of the shortest distance to the sulcus. This parametrization is used in our semi-analytical half-plate model but its computation is not part of this thesis (supplied by supervisor).

the factor $q_0 = 27\,615.1 \text{ Nm}^{-1}$ that makes the total mass distributed on the surface compatible with the mass for point load $V_p = 3 \times 10^8 \text{ Nm}^{-1}$, and x_q as the angular distance from the fracture (in degrees). We need a transformation between x_q and x as we are working with distances in meters, which is the following

$$x_q = \frac{180 x}{r_E \pi}$$

where $r_E = 252 \times 10^3 \text{ m}$ is the radius of Enceladus.

We proceed as before by substituting the function 3.17 into the expressions 2.33-2.36 for $C_i(x)$ and computing the expression for the bending function $w(x)$ numerically using Wolfram Mathematica. As an output, we get the bending function in the shape pictured in figure 3.4 which slightly differs from the one for point load, see comparison in section 3.2.3.

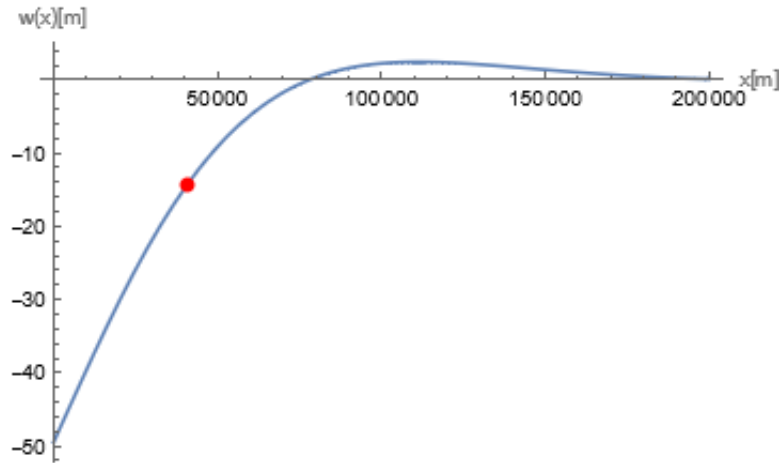


Figure 3.4: Bending function for distributed load with the position of the new crack as a red point

However, the question is whether the position of the new crack exhibits any change for the same physical parameters, see Table 3.1 but different load distribution. We find the answer by computing the maximum bending moment pictured

in Figure 3.5 for $w(x)$ with the realistic load 3.17 that leads us to the position of the new crack at

$$x_m = 40\,550.7 \text{ m.} \quad (3.19)$$

We can see that the distributed load moved the new position of the crack only about a sixth of the original distance further. But since we know that the original position 3.16 corresponds to the real situation on the moon it raises some suspicions about the other physical parameters. After having a quick look at Table 3.1 we can observe that the only parameter that is not a material or the moon's characteristic is the elastic thickness of the ice shell.

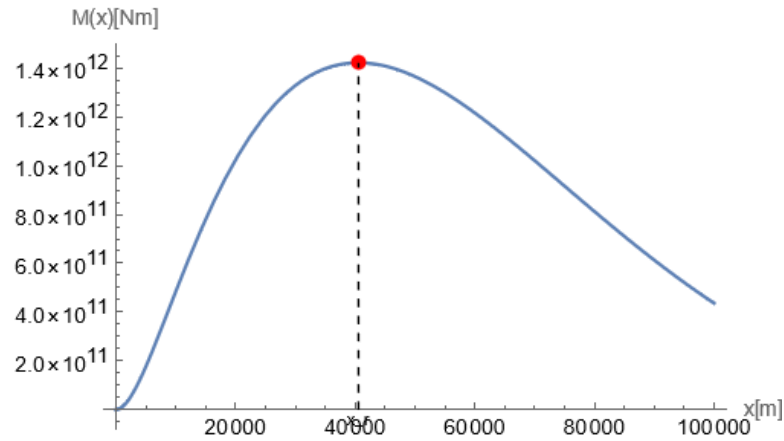


Figure 3.5: Bending moment for distributed load with the position of the new crack as a red point

3.2.3 Conclusion

Coming to the conclusion of the effects of two different load distributions on the surface of the moon on the position of the new crack. First of all, we compared the same amount of mass

$$\int_0^\infty q_d(x)dx = \frac{V_p}{2}$$

but for different distributions. The distribution's effect on the bending function can be seen in Figure 3.6. Speaking in numbers, we calculated that the position of the new crack for the distributed load moved further by approximately 19% compared to the original position for the point load, wherein the amplitude of the bending function, i.e. $w(0)$, decreased by 18% for the distributed load.

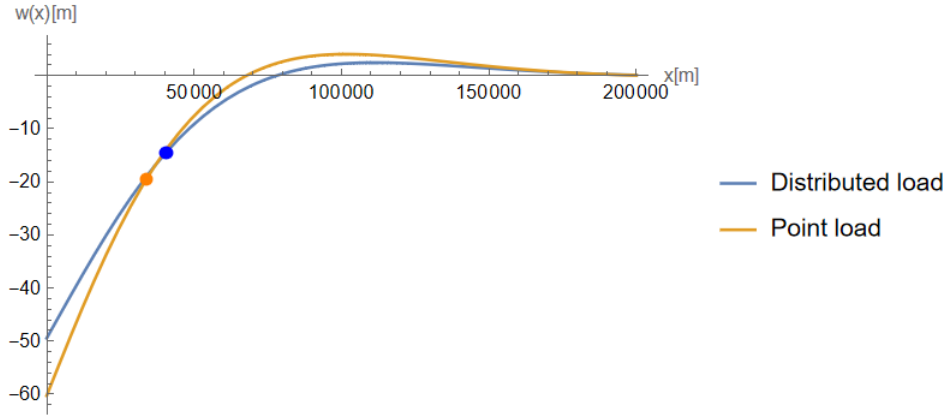


Figure 3.6: Comparison of bending of the plate for point and distributed load

In the next figure, Figure 3.7, we can see that the change in amplitude is much higher, approximately 32% of the original amplitude for the point load. This means that we would need more of the particles accumulated on the surface to reach the critical bending moment which is defined by the relation

$$M_{crit} = \frac{\sigma_{crit} T_e^2}{6}$$

from Hemingway et al. [2020], where $\sigma_{crit} = 1\text{MPa}$ is the tensile failure limit for cold ice.

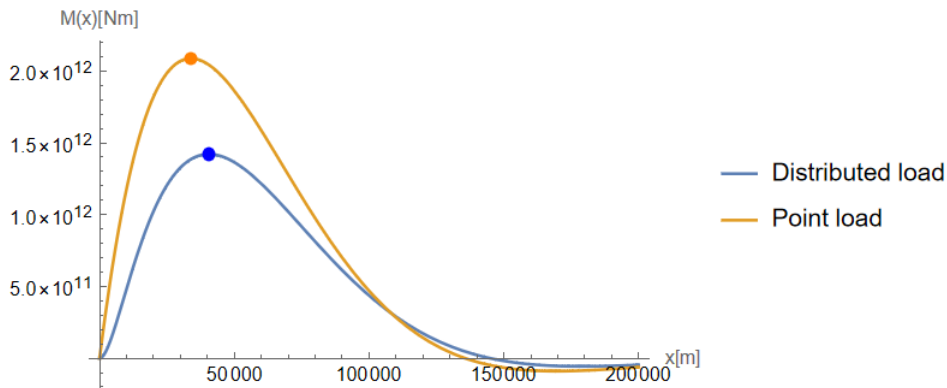


Figure 3.7: Comparison of the bending moments for point and distributed load

4. Discussion and conclusions

Although the more simple approximation of the load distribution on the surface of Enceladus might give us neat results corresponding to the realistic position of the new crack as well as easier computations, the realistic model shows us that the real situation might be different in other aspects of the problem, such as the estimate of the elastic shell thickness. Let us have a look at this question.

The goal is to find the dependence of the elastic shell thickness on the position of the maximal bending moment which can be constrained from observations of the known position of the crack. For point load, it can be computed analytically by expressing the position x_m from 3.11 as a function of the elastic shell thickness T_e and inverting this relationship. We get the dependence by substituting for α defined in 2.5 and D in 1.37, where the T_e is hidden

$$x_m = \alpha \frac{\pi}{4} = \frac{\pi}{4} \left(\frac{4}{\rho g} \right)^{\frac{1}{4}} \left(\frac{ET_e^3}{12(1-\nu^2)} \right)^{\frac{1}{4}}. \quad (4.1)$$

We end up with the expression

$$T_e \approx x_m^{\frac{4}{3}} \quad (4.2)$$

which is plotted as the blue line in Figure 4.1.

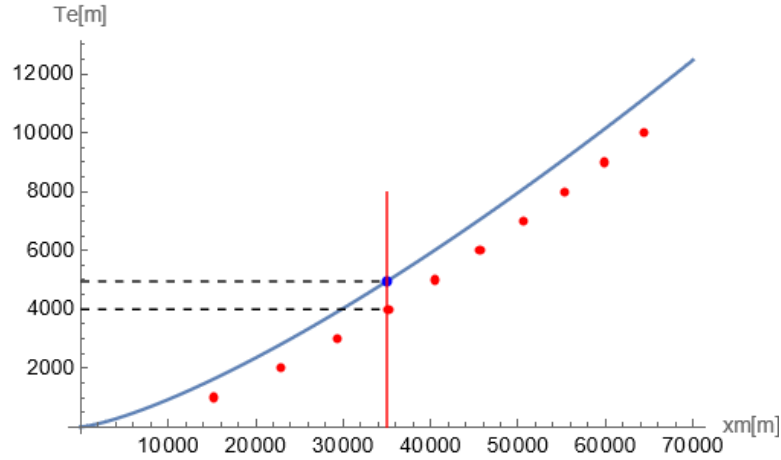


Figure 4.1: The relationship between the position of the new crack depending on the elastic shell thickness. The blue line represents the point load and the red points the distributed load.

For the distributed load, we computed each (red) point separately for T_e from 1-10 km. By plotting these points we can compare the change in the elastic shell thickness for 35 km as the fixed distance between the tiger stripes and get a decrease in the thickness for the distributed load by 19%. This signifies that the distributed load affects the ice crust's thickness, and thus, it is worth working with for future studies as it better represents the real situation (load-wise).

Taking the relationship between the shell thickness d and the elastic shell thickness T_e as $\frac{T_e}{d} \approx 0.59-0.68$ from Hemingway et al. [2020] we can calculate that for the point load the actual thickness of the plate for $T_e = 5$ km is $d \approx 8.3$ km. In comparison, the lower value of T_e we got for the distributed load $T_e = 4$ km would

thus be more compatible with the independent estimates of the shell thickness on the south polar region of Enceladus, reaching values of only a few kilometers (Čadek et al. [2016]).

In conclusion, starting with an approximation of a relatively small moon in diameter by an infinite plate and using the variational principle of virtual displacements to derive the leading equation and the boundary conditions for a complex problem, we arrive at results that not only describe the real situation more closely but will be also useful for further research in the elastic and total shell thickness of the ice crust of Enceladus. We further plan on doing our own computations with the supplied data (Southworth et al. [2019]) and studying the problem in 3D also considering the damage mechanics.

Bibliography

- Ondřej Čadek, Gabriel Tobie, Tim Van Hoolst, Marion Massé, Gaël Choblet, Axel Lefèvre, Giuseppe Mitri, Rose-Marie Baland, Marie Běhouňková, Olivier Bourgeois, et al. Enceladus's internal ocean and ice shell constrained from cassini gravity, shape, and libration data. *Geophysical Research Letters*, 43 (11):5653–5660, 2016.
- Douglas J Hemingway, Maxwell L Rudolph, and Michael Manga. Cascading parallel fractures on enceladus. *Nature Astronomy*, 4(3):234–239, 2020.
- Zdeněk Martinec. Principles of continuum mechanics. *Birkhauser Basel*. doi, 10 (1007):978–3, 2019.
- Victor F Petrenko and Robert W Whitworth. *Physics of ice*. OUP Oxford, 1999.
- Junuthula Narasimha Reddy. *Theory and analysis of elastic plates and shells*. CRC press, 2006.
- Ben S Southworth, Sascha Kempf, and Joe Spitale. Surface deposition of the enceladus plume and the zenith angle of emissions. *Icarus*, 319:33–42, 2019.
- Donald L Turcotte and Gerald Schubert. *Geodynamics*. Cambridge university press, 2002.
- Robert Černý and Milan Pokorný. Matematická analýza pro fyziky 2. 2016.

List of Figures

1	Illustration of Enceladus' plumes in the south polar region	2
2	Fracture formation	2
3	The parallel fractures on the south pole, so-called 'tiger stripes' .	3
1.1	Sketch of the problem geometry	5
3.1	The bending moment and deflection for point load	22
3.2	Decadic logarithm of normalized surface mass deposition rate . . .	23
3.3	Zoomed in decadic logarithm of normalized surface mass deposition and 1D fit of the dataset	24
3.4	Bending function for distributed load	24
3.5	Bending moment for distributed load	25
3.6	Comparison of bending of the plate for point and distributed load	26
3.7	Comparison of the bending moments for point and distributed load	26
4.1	The relationship between the position of the new crack depending on the elastic shell thickness	27

List of Tables

3.1	Table of Physical Parameters	22
-----	--	----