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**Numerical simulation of the wobble of
Venus in response to surface forcing**

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I would like to thank my supervisor Vojtěch Patočka for his invaluable assistance and boundless patience. I truly couldn't do this without him, though I suppose that goes without saying.

Thanks to all the math teachers in my life for in turn awakening and nurturing my passion for exploring.

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Title: Numerical simulation of the wobble of Venus in response to surface forcing

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Abstract: Planetary bodies may change their orientation in space. This may occur due to external torques, which cause precession of the spin axis. Internal processes, such as mantle convection, also have an influence. They change the density structure of the body without affecting the angular momentum. Here, we study planetary reorientation in response to torque exerted on the solid body by the atmosphere surrounding it.

Venus has an extremely slow rotation, and its rotational bulge is thus very small, which amplifies the effect of perturbations. In addition, its atmosphere is extremely dense, and the flows within it exert large torques on the planet.

In this project, I modified an existing numerical code to include surface torques in simulations of true polar wander. I then examine the influence of torques on the movement of the rotation axis in Venus' body-fixed frame. I also track the effect on the angle between the rotation axis and the main axis of inertia, which is abnormally large on Venus.

Keywords: planetary reorientation, numerical modeling, Venus

Název práce: Numerická simulace kolébání Venuše v důsledku povrchových sil

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Abstrakt: Planety mohou měnit svou orientaci v prostoru. To může být způsobeno momenty vnějších sil, které způsobují precesi osy otáčení. Vliv mohou mít i vnitřní děje, které mění rozložení hustoty tělesa, ale nenaklápí osu otáčení. Mezi ty patří například konvekce v plášti. Zde zkoumáme reorientaci planet vlivem momentu sil, kterými na těleso působí atmosféra.

Rotace Venuše je velmi pomalá, což vede k malému rovníkovému vydutí a zvýšenému vlivu perturbací. Její atmosféra je navíc velmi hustá a proudy v ní vyvíjejí na planetu velké momenty sil.

V tomto projektu upravuji stávající numerický kód, aby v simulacích pohybu pólů zahrnoval momenty povrchových sil. Dále zkoumám vliv momentů sil na pohyb osy otáčení v soustavě spojené s tělesem. Mimo to sleduji i vliv na úhel mezi osou otáčení a hlavní osou setrvačnosti, který je u Venuše abnormálně velký.

Klíčová slova: reorientace planet, počítačové modelování, Venuše

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Introduction

The rotation of planets is governed by the law of conservation of angular momentum. With no outside interference, their axes of rotation retain their orientation in an inertial frame. Even without external torques, the planet can reorient, and the position of its rotation axis may change in the body-fixed frame.

This can happen due to mass redistribution altering the inertia tensor (e.g., by tectonic activity, changes in ocean or glacier distribution, or the ever popular meteor impact).

Venus is the slowest rotating planet in the solar system – a Venus day takes 243 Earth days, longer than its orbital period of 225 days, and its equatorial bulge reflects this, being nearly unmeasurable compared to the equatorial bulge of Earth. As such, the inertia tensor of Venus has a very uniform spectrum, so a relatively small perturbation in mass distribution can cause a large change in the direction of eigenvectors.

Sources of planetary reorientation can be divided into internal and external actors. External influences such as tidal forces usually contribute in the form of precession in an inertial frame. This is traditionally not considered true polar wander and is not included in simulations thereof.

True polar wander is classically driven by internal processes, such as the reforming of the equatorial bulge and mantle convection. It is traditionally tracked in the body-fixed frame.

Here we use the numerical tool LIOUSHELL that simulates true polar wander of celestial bodies by solving Liouville’s differential equation,

$$\mathbf{I} \cdot \frac{d\vec{\omega}}{dt} + \left(\frac{d\mathbf{I}}{dt} \cdot \vec{\omega} + \vec{\omega} \times (\mathbf{I} \cdot \vec{\omega}) \right) = \vec{M} \approx 0 \quad (1)$$

for radially stratified viscoelastic bodies.

An interesting quantity to observe is the angular separation between the rotation vector $\vec{\omega}$ and the figure axis (MIA). On Venus, according to measured gravity data, this offset is 0.5° , atypically large in comparison to other planets.

Studies such as [1] have attempted to explain this abnormality with simulations driven by mantle convection. However, it is unlikely that mantle convection alone can drive the observed $\vec{\omega} - MIA$ offset.

Another possible explanation may be external torques. The right side of equation 1, the surface torque, was approximated as 0 in previous simulations. In more typical cases, the influence of external forcings such as tidal forces is periodic and can be neglected when averaged over longer time scales.

In the case of Venus, there are factors such as atmospheric currents that can potentially disrupt the rotation of Venus by exerting a significant torque on its surface. The overall angular momentum is relatively small, so an external torque can effectively alter its direction. More importantly, the difference between the main and secondary inertial axes is very small. Thanks to this, changes in the figure axis direction are much less restricted compared to other planets.

In my simulations, I separate Venus’ atmosphere from the rest of the planet. This leads to a hybrid approach of sorts. The torques exerted by the atmosphere

are technically external, but averaged over a long timescale, their influence is not negligible.

In textbooks, the movement of spinning bodies affected by external torques (forced oscillators) is usually described in an inertial frame. However, the problem of polar wander tracks the rotation vector in a body-fixed frame. This is not very well described for forced rotators, and analytic solutions exist only for specific cases. Here, this phenomenon will be explored numerically, which allows us to expand this work in the future to study an arbitrarily prescribed time evolving forcing from the atmosphere.

To account for external forcings, I have modified LIOUSHELL to accept an external torque as an input and alter the governing equations accordingly. In this thesis, I will analyze the effect of different torque values on polar wander, both as a standalone loading and in conjunction with other driving forces.

1 Motivation

The orientation of planetary bodies in space is not stable, they may reorient with respect to their rotation axis. This can have various causes such as internal or external loadings or external torques. Reorientation leads to changes in the shape of the body itself, because it results in readjustment of the rotational bulge, which controls and feedbacks further reorientation (Figure 1.1).

Figure 1.1 shows the step by step progress of planetary reorientation in response to internal loading.

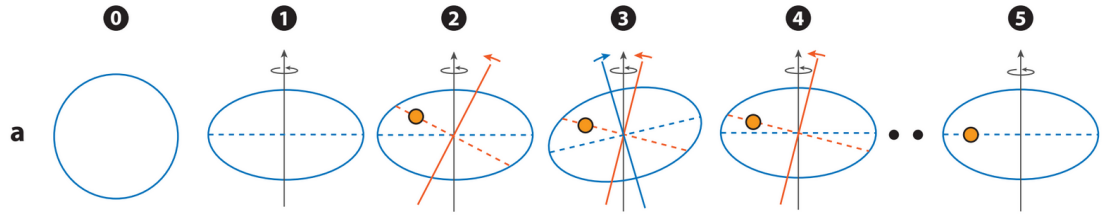


Figure 1.1 A schematic of planet reorientation due to a mass loading, adopted from [2]

The orange circle represents a positive mass load and the black arrow is the rotation vector. The blue and orange arrows represent the main inertia axes of the rotational bulge and that of the load respectively. Inertia tensor represents the density structure of the body and is symmetric. It has three eigendirections, the one with the largest eigenvalue is called the figure axis.

In figure 1.1, panel ①, the planet is in a hydrostatic equilibrium. In panel ②, a perturbation is introduced – a mass load appears, changing the inertia tensor and shifting the figure axis. This causes the axis of rotation to shift in the planet’s body-fixed frame ③. The change in rotation leads to a change in the planet’s energy potential, causing a new bulge to form ④. If the planet’s shell can fully relax into a new equilibrium, the load may eventually shift all the way to the equator ⑤ and the axis of rotation will coincide with the principal axis of the load. [2]

The shape of most planets is primarily dictated by their rotation.

On Earth, the difference between polar and equatorial radii is over 20 km, making the main eigenvector of its inertia tensor very robust. The sources of rotational disturbance on Earth are very small compared to the mass of its bulge. Currently, the angular distance of Earth’s rotation vector from its main inertial axis is less than $0.3''$ and it is estimated that on geological time scales, these axes have remained similarly close to each other.

In comparison, the slow rotation of Venus creates an equatorial bulge of only about 30 cm. Such a small bulge infers a difference between the main and minor moments of inertia, $C - A$, of around $3 \cdot 10^{30} \text{kg} \cdot \text{m}^2$, which is smaller than the $C - A$ value obtained from gravity data by a factor of 300.

Therefore, the inertia tensor of Venus must be dominated by features other than rotation, a likely candidate being mantle convection and the associated

dynamic topography.[3]

Here, the reorientation of planets is tracked in terms of polar motion – the position of $\vec{\omega}$ over time in a body-fixed frame.

Causes of polar wander can be divided into internal and external drivers. Internal actors include the planet’s gravitational pull on itself, stresses in the lithosphere or for example flow of magma in the mantle.

Since Spada et al. 1996[4], mantle convection has been investigated as a driving force for true polar wander on Venus.

In this thesis, I will be focusing on externally driven polar wander. Internal processes will only serve as a reference to compare against. External factors acting on a typical planet consist mostly of tidal forces. However, in planets that are not tidally locked, their influence is periodic and can usually be neglected over long time scales.

The atmosphere of Venus is extremely dense with a pressure around 9MPa. The drag from major air currents generates non-negligible torques on the order of $10^{20}\text{kg} \cdot \text{m}^2/\text{s}^2$ [5]. It is speculated that the long-term influence of atmospheric drag could even be the cause of Venus’ retrograde rotation.

In this thesis, I investigate the effect of torques due to forces applied to Venus’ surface upon polar wander.

For my model, the atmosphere is considered as an external actor, which can exert forces on the surface. This creates a torque in Liouville’s equation – the conservation of angular momentum expressed for a variable tensor of inertia.

2 Theory

2.1 Introduction

One of the main governing principles of polar wander is the law of conservation of angular momentum.

$$\vec{M}_i = \frac{d\vec{H}_i}{dt} + \varepsilon_{ijk}\vec{\omega}_j\vec{H}_k \quad (2.1)$$

where $\vec{M}$ is the applied torque, $\vec{H} = I(t)\vec{\omega} + \vec{h}(t)$ is the angular momentum, $\mathbf{I}(t)$ is the time-dependent tensor of inertia and $\vec{h}_i(t) = \int_V \rho \varepsilon_{ijk} x_j \vec{u}_k dv$ is the angular momentum due to deformation $\vec{u}$ observed in the frame of reference.[6]

Substituting for $\vec{H}$ we obtain,

$$\vec{M}_i = \frac{d}{dt}(\mathbf{I} \cdot \vec{\omega} + \vec{h})_i + \varepsilon_{ijk}\vec{\omega}_j(\mathbf{I} \cdot \vec{\omega} + \vec{h})_k \quad (2.2)$$

Equation (2.2) is called the Liouville equation (LE), differing from the classical mechanics of rigid bodies in that the body can deform, which yields the relative angular momentum $\vec{h}$.

Despite this conceptual difference, a reference frame can always be chosen such that $\vec{h}$ is equal to zero (Tisserand frame [6]). If we consider the problem in this frame, the equation simplifies to the form identical to the conservation of angular momentum of rigid bodies:

$$\vec{M} = \frac{d\mathbf{I}}{dt} \cdot \vec{\omega} + \mathbf{I} \cdot \frac{d\vec{\omega}}{dt} + \vec{\omega} \times (\mathbf{I} \cdot \vec{\omega}) \quad (2.3)$$

In general, polar wander is the solution of the Liouville equation in the Tisserand frame, with $\mathbf{I}$ given by the planet's density distribution.

In the absence of external torques, polar motion can be decomposed into a secular motion and free wobble. For small excursions of the rotation pole, both of these solutions can be obtained analytically by solving a linearized version of the Liouville equation ([6]). The behavior of a forced spinning body is also well known, but it is typically described only in the inertial frame of reference, while here the focus is on the solution in the body-fixed frame.

Numerical simulations allow us to consider a large $\Delta\vec{\omega}$ and a non-constant external torque. As such, linearized approximations would not be accurate and I couldn't cover the full range of cases with an analytic approach.

2.2 Precession and nutation

This thesis is focused on torques resulting from atmospheric drag. For the sake of completion, let us first introduce torques exerted by surrounding bodies. "As a planet's rotation axis is tilted with respect to the orbital plane, the equatorial bulge is out of the equatorial plane during the orbital motion. As a result, the Sun and moon(s) exert a gravitational torque on the planet, tending to twist the equator towards the orbital plane of the Sun/moon relative to the planet. As the planet is rotating, it reacts to this torque like a spinning top to the gravitational

pull on it. The main effect is precession, which is the slow motion of the rotation axis of the planet around the normal to its orbital plane."

The precessional and nutational motions thus result from torques by the gravitational potentials of the moon(s) and the Sun (and of the planets, to a minor extent). [7]

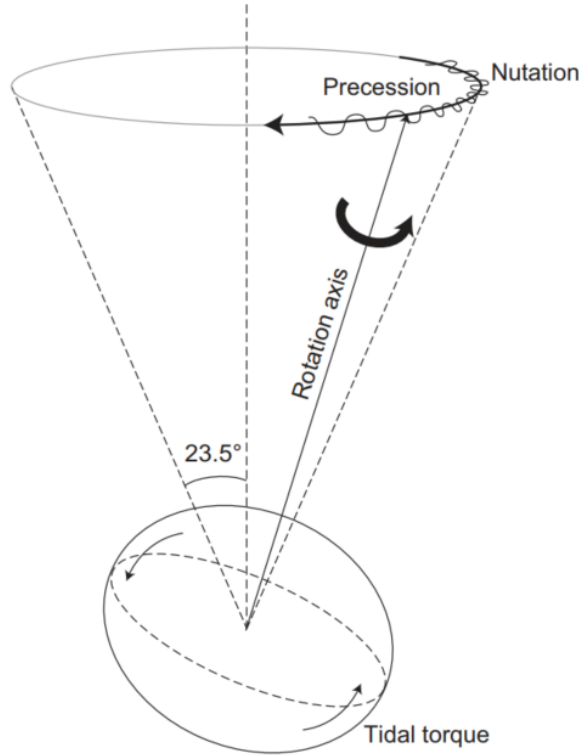


Figure 2.1 Precession and nutation (on Earth). Precession is a smooth motion of the pole of the Earth's axis along the circle around the normal to the ecliptic plane (shown by the dashed vertical line), and the wiggly excursions from the precessional path represent nutation.[7]

"The term wobble is used in a broad sense for any periodic or quasi-periodic motion of the planet irrespective of the frequency or the physical origin of the motion. The essence of wobble is the separation of the rotation axis (i.e., the direction of the instantaneous angular velocity vector) from the direction of the mean angular velocity vector Ω . Wobble is characterized by the equatorial components (m_1, m_2) of (Ω/Ω_0) . (Variations of m_3 represent the fractional variations in the spin-rate, which manifest themselves in length-of-day variations.) Wobble is an inescapable accompaniment of nutation; it is, in fact, intimately related to the nutation of the figure axis, and plays an essential role in various approaches to the treatment of nutation." [7]

During precession, the rotation axis may significantly change its direction in the inertial frame of reference, but it may be accompanied with only a negligible change of position of the rotation pole in the body-fixed frame. In the body-fixed frame, gravitational torque from the Sun and Moon changes its direction diurnally, while in this work the torque stays time constant in the body-fixed frame (with the exception of Sections 4.2 and 4.3).

2.3 Force-free wobble

One problem in dynamics where Liouville's equation is applicable is in the motion of a body not subject to any net torques. Forces that do not exert a torque do not affect the rotation of a body, and it does not decrease the generality of the solution to discuss the rotational motion in a reference frame in which the center of mass is fixed. In such a case, the angular momentum arises only from rotation about the center of mass, and Liouville's equation is the equation of motion for the complete system. In the absence of any net torques, it reduces to

$$\mathbf{I}_1 \dot{\vec{\omega}}_1 = \vec{\omega}_2 \vec{\omega}_3 (\mathbf{I}_2 - \mathbf{I}_1) \quad (2.4)$$

$$\mathbf{I}_2 \dot{\vec{\omega}}_2 = \vec{\omega}_1 \vec{\omega}_3 (\mathbf{I}_3 - \mathbf{I}_2) \quad (2.5)$$

$$\mathbf{I}_3 \dot{\vec{\omega}}_3 = \vec{\omega}_1 \vec{\omega}_2 (\mathbf{I}_1 - \mathbf{I}_3) \quad (2.6)$$

The same equations, of course, will also describe the motion of a rigid body when one point is fixed and there are no net applied torques. We know two immediate integrals of the motion, for both the kinetic energy and the total angular momentum vector must be constant in time. [8] Even the rotational axes of "well-behaved" force-free spinning bodies exhibit periodic motion.

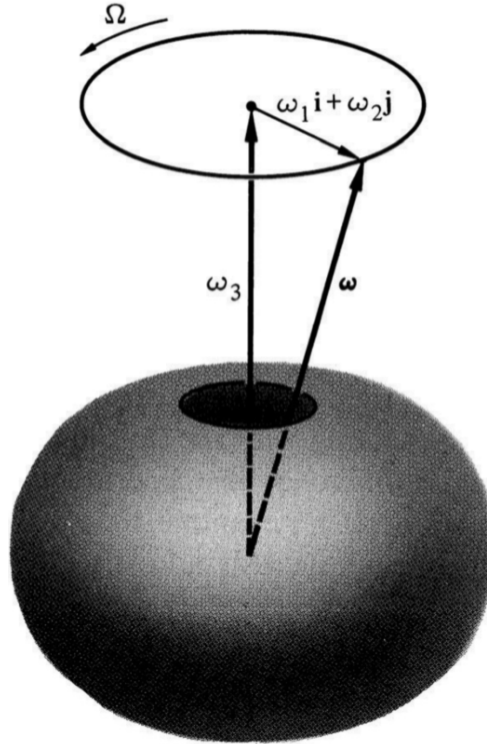


Figure 2.2 Periodic movement of the angular velocity about the axis of symmetry [8]

2.4 Inertia tensors

Consider a body rotating about a fixed axis passing through the origin with the angular velocity $\vec{\omega}$. Let $\rho(\vec{x})$ be the local density of the body at the point $\vec{x}$.^[9] Then, the total angular momentum of the body is

$$\vec{H} = \int_{\vec{x} \in B} \vec{x} \rho(x) \times \frac{d\vec{x}}{dt} dv \quad (2.7)$$

where dv is an infinitesimal element of volume and B is the body.

This can be rewritten in matrix form as

$$\vec{H} = \mathbf{I} \cdot \vec{\omega} \quad (2.8)$$

where $\mathbf{I}$ is a 3×3 real matrix with elements

$$\mathbf{I}_{ii} = \int_{\vec{x} \in B} \sum_{j \neq i} j^2 \rho(\vec{x}) dv \quad (2.9)$$

$$\mathbf{I}_{ij} = - \int_{\vec{x} \in B} x_i x_j \rho(\vec{x}) dv \quad (2.10)$$

Note that in general, $\vec{H}$ is not parallel to $\vec{\omega}$.

2.5 Principal axes

$\mathbf{I}$ is a real symmetric 3×3 matrix. Therefore, the inertia tensor has 3 orthogonal eigenvectors associated with real eigenvalues.

The directions of these are denoted the principal axes of rotation.

$\vec{H}$ is parallel to $\vec{\omega}$ if and only if $\vec{\omega}$ is aligned with one of the principal axes (or a linear combination of principal axes with the same eigenvalue).

Rotation aligned with the main axis of inertia is stable. Otherwise, reorientation is likely. In an extreme case where $\vec{\omega}$ is near a secondary principal axis, the "tennis racket effect" can occur, completely realigning the body. This effect is not yet documented in bodies as large as planets.

3 Numerical method

3.1 LIOUSHELL

LIOUSHELL is a general Liouville equation solver for viscoelastic planetary shells. [10]

Polar motion is simulated by solving two sets of coupled equations.

The first set governs the viscoelastic response to changes in the direction and speed of rotation, that is, it describes the time evolution of the rotational bulge.

The second set is the Liouville equation itself, which takes the deformation of the body as an input and determines the evolution of $\vec{\omega}$ over time as an output. [10]

The viscoelastic response of a hydrostatically prestressed, radially stratified shell is obtained by integrating the following set of differential equations:

$$\nabla \cdot \tau + \vec{f} = 0 \quad (3.1)$$

$$\nabla \cdot \vec{u} = 0 \quad (3.2)$$

$$\tau^d - \mu(\nabla \vec{u} + (\nabla \vec{u})^T) = -\frac{\mu}{\eta} \int_0^t \tau^d dt' \quad (3.3)$$

the conservation of momentum 3.1, the incompressibility condition 3.2, and Maxwell's equation of viscoelastic rheology 3.3. [10]

τ is the Cauchy stress tensor, τ^d denotes its deviator, t is time, $\vec{u}$ is the displacement vector, μ is the elastic shear modulus and η is the viscosity. $\vec{f}$ is the total body force.

At the time $t = 0$, the right-hand side of the third equation is 0 and the latter two equations govern the initial elastic response.

To compute the evolution of $\vec{\omega}$, the Liouville equation is solved:

$$\mathbf{I} \cdot \frac{d\vec{\omega}}{dt} = -\frac{d\mathbf{I}}{dt} \cdot \vec{\omega} - \vec{\omega} \times (\mathbf{I} \cdot \vec{\omega}) + \vec{M} \quad (3.4)$$

$$\frac{d\vec{\omega}}{dt} = -\mathbf{I}^{-1} \cdot \left(\frac{d\mathbf{I}}{dt} \cdot \vec{\omega} + \vec{\omega} \times (\mathbf{I} \cdot \vec{\omega}) - \vec{M} \right) \quad (3.5)$$

where $\mathbf{I}$ is the time-dependent inertia tensor and $\vec{M}$ is the external torque.

The Liouville equation is the conservation of angular momentum $\vec{H} = \mathbf{I} \cdot \vec{\omega}$ expressed in the body-fixed Tisserand frame. Equations 3.1–3.3 are coupled with LE through the displacement field (which is used to calculate the inertia tensor) and through the centrifugal potential that contributes to the body forces (and depends on $\vec{\omega}$).

$\mathbf{I}$ is calculated with the MacCullagh formula, using the gravitational potential and the initial density profile [10].

3.2 Implementing surface torque

In my implementation, the atmosphere is considered a system separate from the planet itself. As such, the effect of atmospheric friction is included in the form of an external torque $\vec{M}$ that is inserted directly into the Liouville equation (see Chapter 3 in [6] for alternative options, such as to account for the relative angular momentum of the atmosphere instead of the surface force it exerts).

In practice, the tensor quantities used in numerical calculations are scaled by scalars of appropriate orders of magnitude to reduce rounding errors. The Liouville equation is then divided by these scalars:

$$\omega_0 \frac{d\hat{\vec{\omega}}}{dt} = \frac{\hat{\mathbf{I}}^{-1}}{I_0} \cdot \left[- \left(I_0 \frac{d\hat{\mathbf{I}}}{dt} \cdot \omega_0 \hat{\vec{\omega}} + \omega_0^2 I_0 \hat{\vec{\omega}} \times (\mathbf{I} \cdot \hat{\vec{\omega}}) \right) + \vec{M} \right] \quad (3.6)$$

$$\frac{d\hat{\vec{\omega}}}{dt} = -\hat{\mathbf{I}}^{-1} \cdot \left(\frac{d\hat{\mathbf{I}}}{dt} + \omega_0 \hat{\vec{\omega}} \times (\mathbf{I} \cdot \hat{\vec{\omega}}) \right) + \frac{\hat{\mathbf{I}}^{-1} \cdot \vec{M}}{I_0 \omega_0} \quad (3.7)$$

where ω_0 and I_0 are scalars of the same order of magnitude as $\vec{\omega}$ and $\mathbf{I}$ and $\hat{\mathbf{I}}$ and $\hat{\vec{\omega}}$ are the non-dimensional quantities. Therefore, the applied torque must also be scaled accordingly.

Atmospheric currents can exert a significant force on the solid surface of Venus. In the future, the friction can be obtained from a global climate model. Here, it is prescribed rather arbitrarily. As a reference value, the mountain wave effect studied by Navarro et al. (2018) [5] is used. Therein, the peak value of the torque is estimated to be on the order of $10^{20} \text{kg} \cdot \text{m}^2/\text{s}^2$.

Implemented as described above, the torque is prescribed as a time constant vector in the body-fixed frame (Section 4.1). However, in the body-fixed frame, the axis of rotation changes over time. Since torques on Venus studied here are mainly generated by atmospheric flow whose spatial orientation is linked with the planet's rotation, it may be more appropriate to consider a time evolving torque, whose direction reflects the instantaneous position of the rotation pole (Section 4.2).

An arbitrary torque can be prescribed as a surface force applied at a point $\vec{s}$ defined in a geographic frame (tied to the immediate axis of rotation). At this point, the force will have a meridional (pole-to-pole) and zonal (aligned with parallels) component $\vec{F}_M$ and $\vec{F}_Z$. In section 4.1, these components are set at the time $t = 0$ and remain constant in the body-fixed frame. In section 4.2, the components evolve in time so that the meridional component remains aligned with the immediate meridians and the zonal component remains parallel with the immediate equator.

I define a unit vector $\vec{s}$ with the spherical coordinates (θ, ϕ) in a frame aligned with the immediate axis of rotation. This will represent a point on the surface of Venus.

$$\vec{s} = \begin{pmatrix} \sin \theta \cdot \sin \phi \\ \sin \theta \cdot \cos \phi \\ \cos \theta \end{pmatrix} \quad (3.8)$$

Now, I can transform the meridional and zonal components into the body-fixed

frame:

$$\vec{F}_Z = \|\vec{F}_Z\| \vec{s} \times \frac{\vec{\omega}}{\|\vec{\omega}\|} \quad (3.9)$$

$$\vec{F}_M = \|\vec{F}_M\| \vec{s} \times \frac{\vec{F}_Z}{\|\vec{F}_Z\|} \quad (3.10)$$

And from these, I calculate the torque:

$$\vec{M} = R(\vec{s}) \vec{s} \times (\vec{F}_Z + \vec{F}_M) \quad (3.11)$$

where $R(\vec{s})$ is the radius of Venus at $\vec{s}$.

If torque is instead defined directly by its components, 3.11 may be rewritten as

$$\vec{M} = \|\vec{M}_Z\| \vec{s} \times \frac{\vec{F}_Z}{\|\vec{F}_Z\|} + \|\vec{M}_M\| \vec{s} \times \frac{\vec{F}_M}{\|\vec{F}_M\|} \quad (3.12)$$

where $\|\vec{M}_Z\|$ and $\|\vec{M}_M\|$ are components of the surface torque corresponding to the zonal and meridional components of the applied force.

4 Results

4.1 Time constant torque

In this section, the point $\vec{s}$, where surface force is acting, is chosen near the pole ($\theta = 1^\circ$, $\phi = 50^\circ$) and a time constant value of $\vec{F}_M$ is prescribed. This results in a torque perpendicular to the initial axis of rotation (a parallel one would primarily result in length-of-day changes). The torque amplitude $||M_M||$ is varied from 10^{18} to $10^{20} \text{ kg}\cdot\text{m}^2/\text{s}^2$.

In figure 4.2, the rotation vector $\vec{\omega}$ moves along a roughly straight line before turning and slowing down, coming to a rest after a total shift of about 80° . In addition, we see that the angular separation $\omega - MIA$ between the axis of rotation and the main axis of inertia quickly increases to a maximum, then decreases with an exponential-like trend. The value at this peak increases with the torque amplitude, but it is not directly proportional. In figure 4.1, I track the relationship between the maximum value of $\omega - MIA$ and the applied torque.

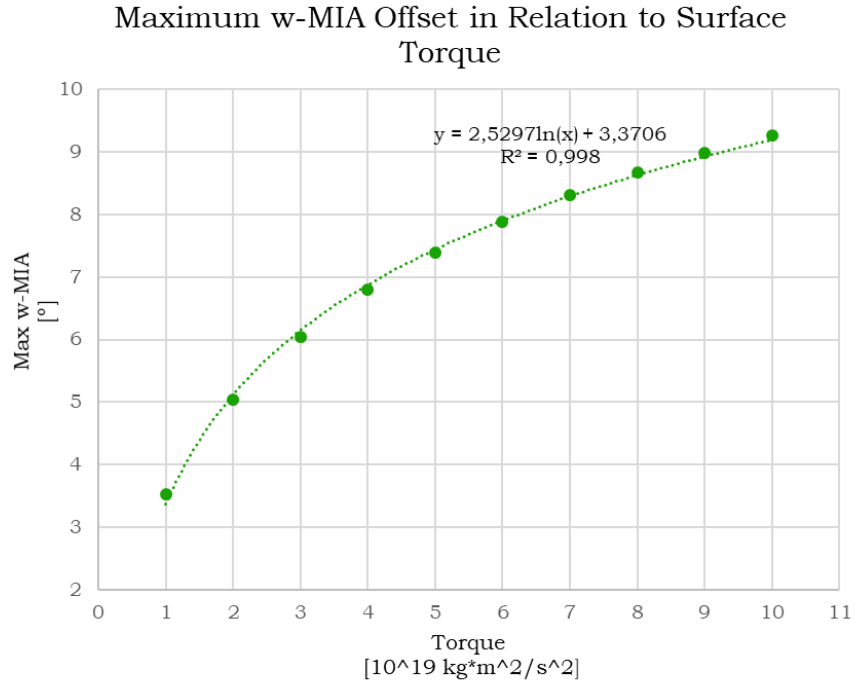


Figure 4.1 Maximum value of the angular offset between the spin and figure poles, w-MIA, in relation to the surface torque amplitude.

In 4.1, the maximum value of $\vec{\omega} - MIA$ for a given torque amplitude is tracked for $||M_M||$ from 10^{19} to $10^{20} \text{ kg}\cdot\text{m}^2/\text{s}^2$ in increments of $10^{19} \text{ kg}\cdot\text{m}^2/\text{s}^2$. The relation between the torque amplitude and the resulting maximum $\vec{\omega} - MIA$ offset appears to be mainly logarithmic for torque amplitudes between 10^{19} and $10^{20} \text{ kg}\cdot\text{m}^2/\text{s}^2$.

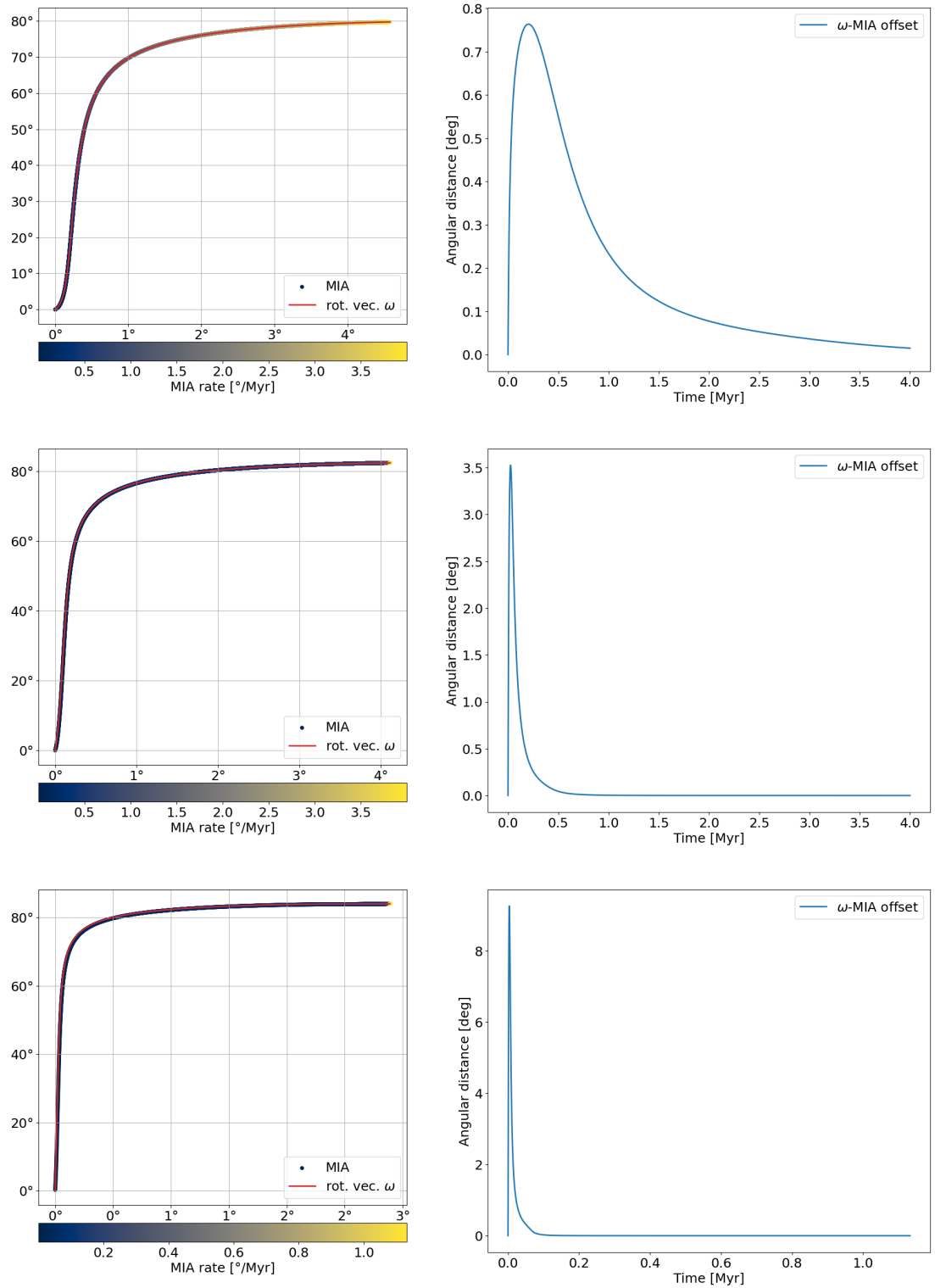


Figure 4.2 Constant torque in the body-fixed frame, amplitude [$\text{kg}\cdot\text{m}^2/\text{s}^2$]: 10^{18} (top), 10^{19} (middle) and 10^{20} (bottom). The graphs on the left track the motion of the rotation vector $\vec{\omega}$ (red curve) and the main axis of the inertia tensor MIA (color-changing curve) in polar coordinates. The graph on the right shows the time evolution of the angular offset between $\vec{\omega}$ and MIA .

4.2 Torque evolving in time

In the previous section, the applied torque is a time constant vector in the body-fixed frame. However, as discussed above, the direction of atmospheric flow depends on the position of the rotation pole, which changes in the body-fixed frame. Therefore, in this section, a time evolving torque is assumed. Here, the surface forces $\vec{F}_Z$ and $\vec{F}_M$ are given by equations 3.9 and 3.10, i.e. they remain zonal and meridional even during the ongoing true polar wander. Similarly as in the previous section, only $\vec{M}_M$ is considered, while $\vec{M}_Z$ is assumed to be zero.

In figure 4.3, at a torque of $10^{17} \text{ kg}\cdot\text{m}^2/\text{s}^2$, the trajectories of $\vec{\omega}$ and MIA are qualitatively similar to those in 4.2 and $\vec{\omega} - MIA$ grows logarithmically. For torque amplitudes of $10^{18} \text{ kg}\cdot\text{m}^2/\text{s}^2$, both $\vec{\omega}$ and MIA travel along an outward spiral. Qualitatively, it seems that for all examined torques from $10^{18} \text{ kg}\cdot\text{m}^2/\text{s}^2$, a similar effect is occurring at different rates: $\vec{\omega} - MIA$ is initially dominated by a sinusoidal oscillation, which eventually decays. Afterwards, the offset trends upward to a maximum before starting to decrease. Interestingly, for the torque amplitude 10^{20} , the maximum $\vec{\omega} - MIA$ offset is close to the value generated by an equivalent body-constant torque.

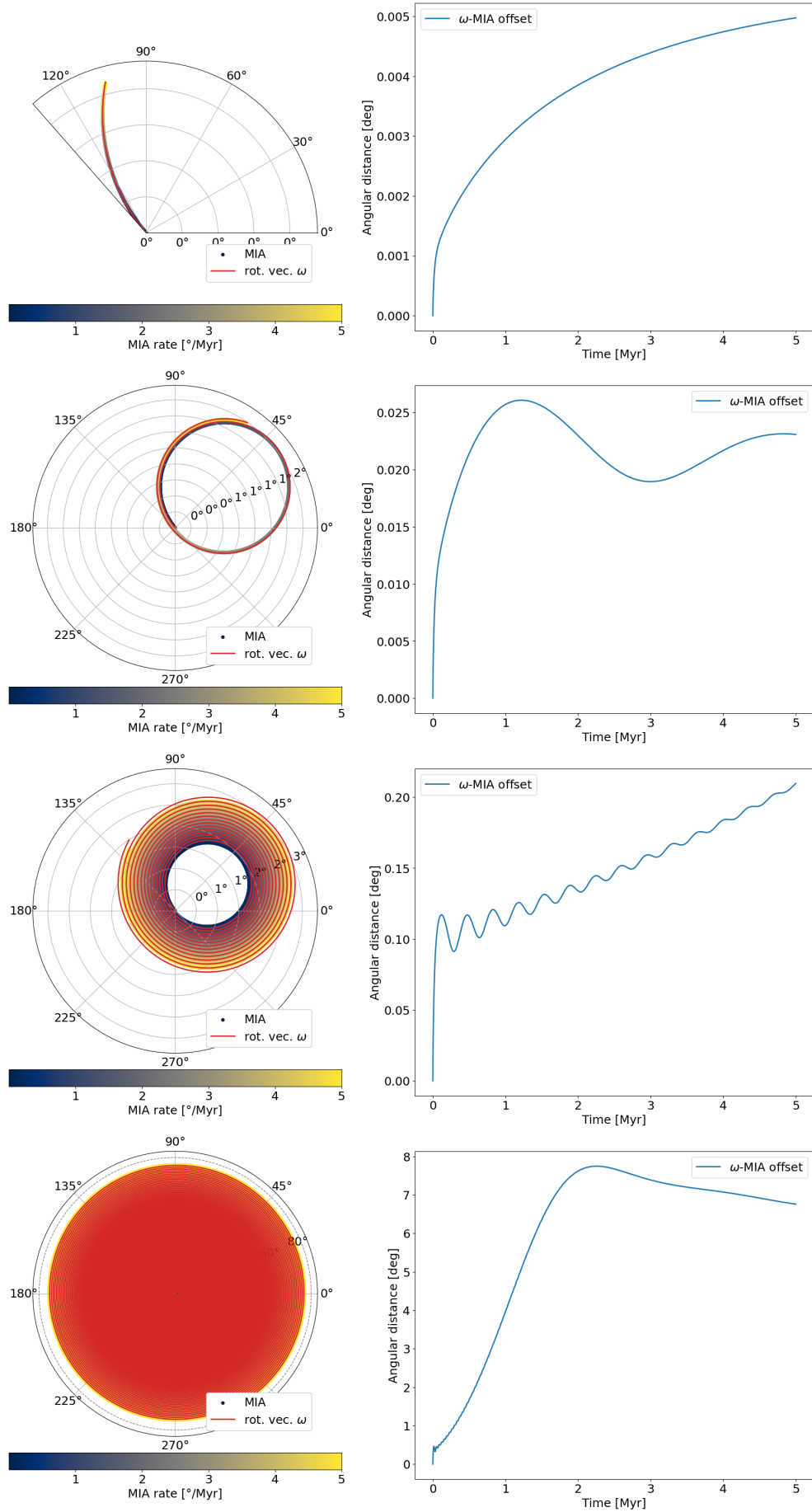


Figure 4.3 Constant torque in a geographic frame, amplitude $[\text{kg}\cdot\text{m}^2/\text{s}^2]$ (top to bottom): 10^{17} , 10^{18} , 10^{19} and 10^{20}

4.3 Grafting onto mantle convection

In this section, surface torque is combined with internal loading. The internal loading is simulated through a time evolving, artificially prescribed Inertia tensor $\mathbf{I}_{MC}(t)$ that is added to the total inertia tensor of the body $\mathbf{I}$ and whose main axis is plotted in color in figure 4.4 (referred to as MID-MC). In figure 4.4, the reference case of this section is shown, where polar wander is driven solely by the internal loading. The reference case is followed by simulations in which atmospheric torque is added, in order to see how $\vec{M}$ acts in combination with internally driven polar wander. Due to non-linearity of the Liouville equation, one can expect that the solutions will not be a mere superposition of the individual contributors.

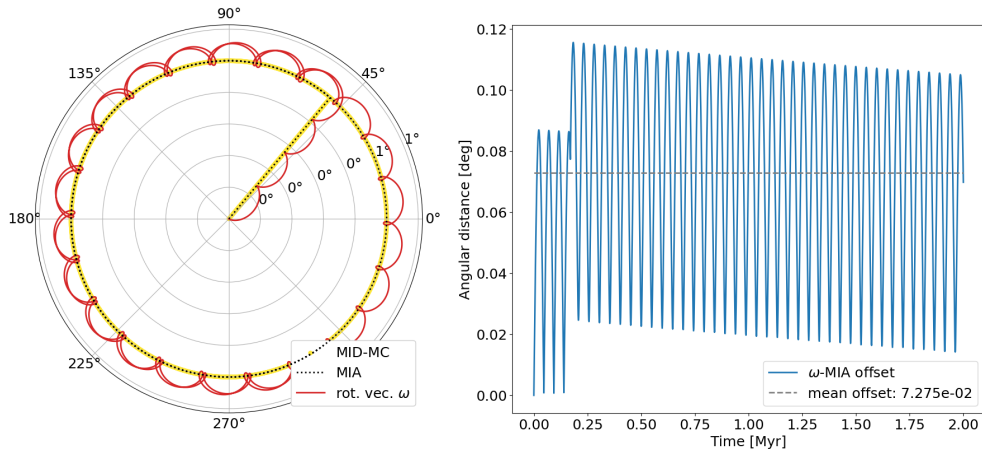


Figure 4.4 Torque-free reference case driven only by an artificially prescribed inertia contribution $\mathbf{I}_{MC}(t)$, mimicking e.g. mantle convection. Parameters describing the $\mathbf{I}_{MC}(t)$ components can be found in [11]. The rotation vector oscillates to and from the MID-MC, with one cycle being equal to the Chandler period.

In figure 4.4, $\vec{\omega}$ starts its trajectory at a right angle with respect to the main inertial direction of $\mathbf{I}_{MC}(t)$ ($MID - MC$), turning back towards it periodically. $MID - MC$ moves in a straight line from its initial direction, then travels along a parallel at the colatitude of 1° . The $\vec{\omega} - MIA$ offset obtained in this simulation is about 0.1° . That is smaller than the measured offset by a factor of 5.

In figure 4.5, time evolving torque is applied as in 4.2 and studied in combination with the internal loading. At $10^{18} \text{kg} \cdot \text{m}^2/\text{s}^2$, the distance by which $\vec{\omega}$ deviates from $MID - MC$ changes with a period about equal to that of $\vec{\omega}$ circling its original position. $\vec{\omega} - MIA$ oscillates between 0 and 0.15° . At $10^{19} \text{kg} \cdot \text{m}^2/\text{s}^2$, this "slow oscillation" grows in amplitude, becoming the dominant observable movement. $\vec{\omega} - MIA$ is around 0.25° . At $10^{20} \text{kg} \cdot \text{m}^2/\text{s}^2$, the oscillation has completely overtaken $\vec{\omega}$'s original trajectory. $\vec{\omega} - MIA$ grows to 15° .

In figure 4.6, constant torque is applied in the body-fixed frame. At a torque of $10^{18} \text{kg} \cdot \text{m}^2/\text{s}^2$, $\vec{\omega}$ moves along an epicycle that is offset from the trajectory of $MID - MC$. $\vec{\omega} - MIA$ is about 1° , one order of magnitude larger than in the reference case with internal loading only. At $10^{18} \text{kg} \cdot \text{m}^2/\text{s}^2$, the original orbit is suppressed and $\vec{\omega}$ moves along a slowly shifting circle. $\vec{\omega} - MIA$ is about 10° . At $10^{18} \text{kg} \cdot \text{m}^2/\text{s}^2$, the circle's diameter quickly grows and its center starts moving. $\vec{\omega} - MIA$ averages 30° .

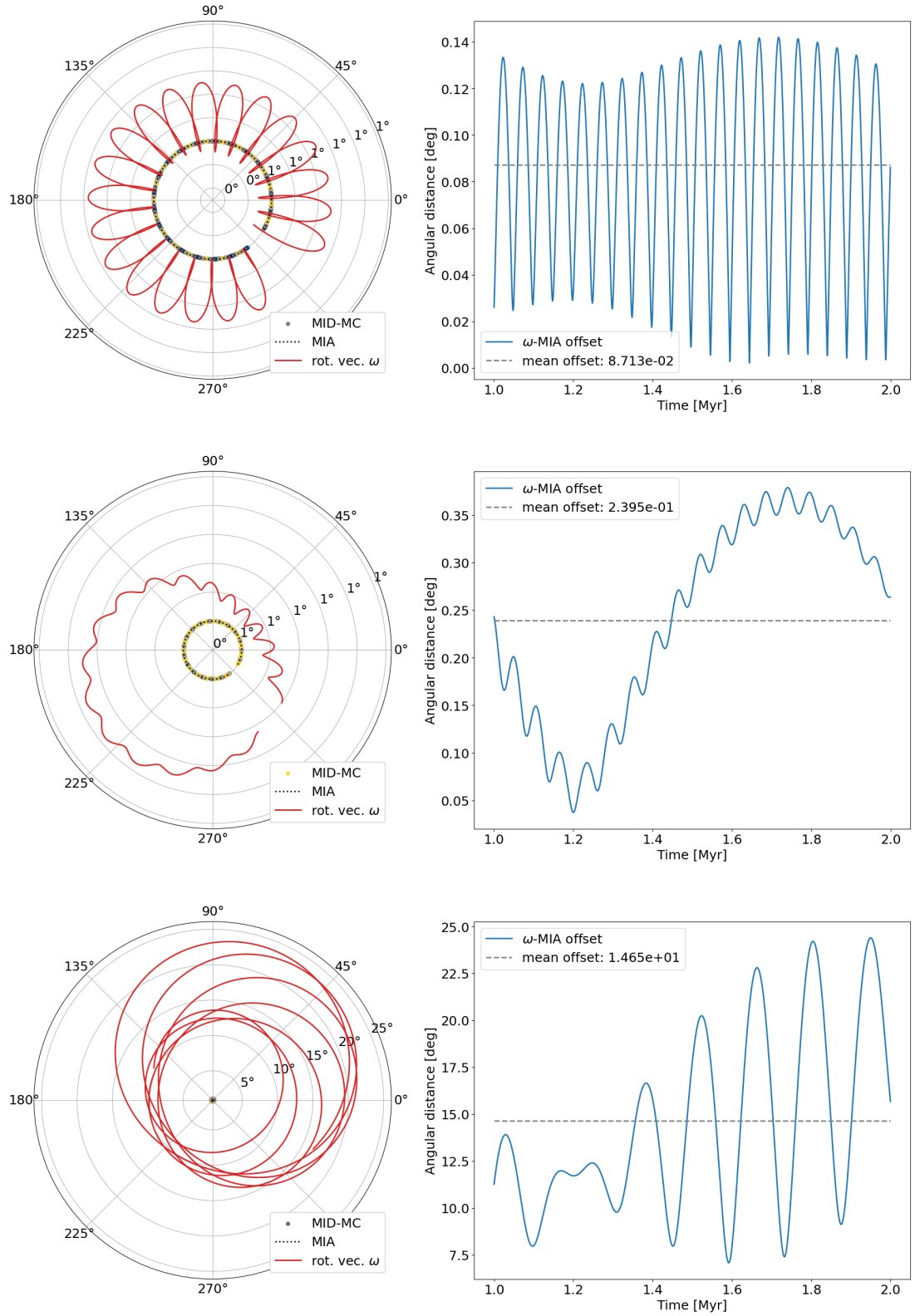


Figure 4.5 Mantle convection coupled with a constant torque in the geographic frame, amplitude [$\text{kg} \cdot \text{m}^2/\text{s}^2$]: 10^{18} (top), 10^{19} (middle) and 10^{20} (bottom)

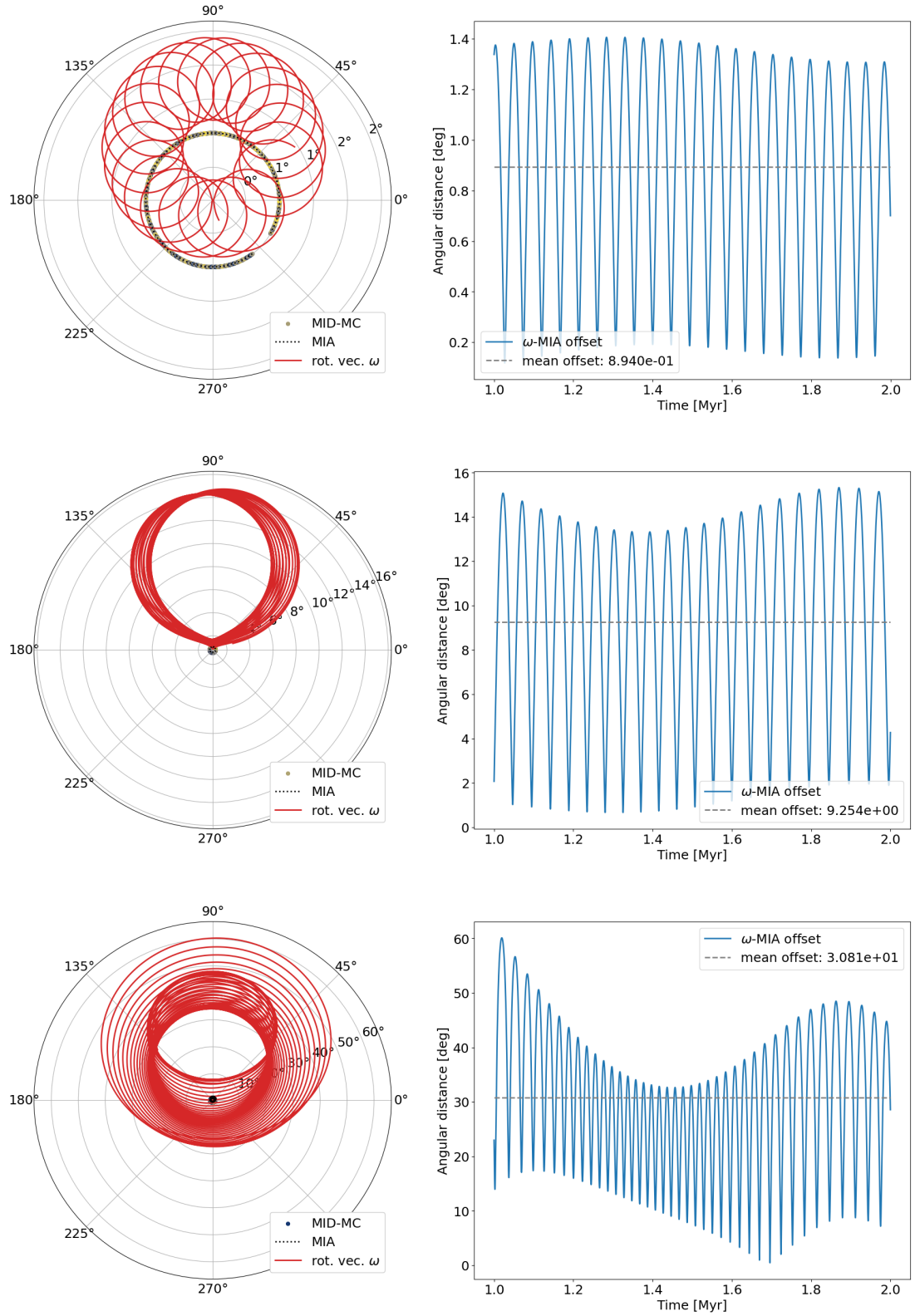


Figure 4.6 Mantle convection coupled with a body-fixed constant torque, amplitude $[\text{kg} \cdot \text{m}^2/\text{s}^2]$: 10^{18} (top), 10^{19} (middle) and 10^{20} (bottom)

Conclusion

In the first part of the thesis, I introduce the issue of the wobble of Venus at a surface level. I describe how Venus differs from more typical planets – its slow rotation and the corresponding shape. I explain how Venus’ dense atmosphere can exert significant torques on the planet. It’s possible that these torques could contribute to the large angular offset between Venus’ axis of rotation and main axis of inertia.

In Chapter 2, I focus on the theoretical aspect of my project. I present a derivation of the Liouville equation from the law of balance of angular momentum and how it governs the polar motion of planets. I describe the behavior of a spinning body with no driving factors.

I give examples of internal and external actors that influence the rotation of a planet. Models of true polar wander traditionally only consider internal processes, such as mantle convection, since external actors (e.g., tidal forces) are in most cases negligible if averaged over a rotational or orbital period.

In this project, the atmosphere is considered a separate system from Venus itself. However, unlike other external sources, the frictions generated by motion of the atmosphere are generally non-zero, even when averaged over long time scales. Therefore, significant forces are exerted on the planet’s surface.

To include these surface forces, I create a hybrid model, which takes into consideration external torques acting on the system. Such a model is unconventional, and its behavior is not described in literature for the large offsets found on Venus. The purpose of this thesis is to explore the model’s behavior via numerical experiments.

In Chapter 3, I describe the numerical method used in this thesis. I introduce the LIOUSHELL solver and the systems of equations it works with. I explain how I modified the program to account for external torques, both in the body-fixed frame and in a frame tied to the immediate axis of rotation.

In chapter 4, I test my method and examine the results yielded by different simulations. In particular, I discuss simulations loaded only with torque and with torque grafted onto mantle convection.

Overall, the results obtained from this project show promise. When the applied torque was comparable to the estimated torques from the mountain wave effect [5], the resulting angular offset between the axis of rotation and the main axis of inertia was comparable to 0.5° , the value obtained from gravity data.

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