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**Influence of Porosity on Tidal
Deformation in Planetary Bodies**

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Title: Influence of Porosity on Tidal Deformation in Planetary Bodies

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Abstract: The thesis describes the derivation of the poroelastic model for the Enceladus core and its subsequent numerical implementation under the assumption that the core is a two-component mixture. The model is described by a set of governing equations based on constitutive relations and physical laws. Strong and weak formulations are derived with the addition of appropriate boundary conditions, and the system is then discretised in time and space. The model and its code implementation are validated against a suitably chosen benchmark problem. Finally, the model's applications to tidal deformation are studied to show the effect of specific poroelastic material constants on the tidal force and deformation in comparison to the purely elastic setting.

Keywords: tidal deformation, porosity, poroelasticity, numerical modelling

Název práce: Vliv porozity na slapovou deformaci planet a měsíců

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Abstrakt: Práce popisuje odvození poroelastického modelu pro jádro Enceladu a jeho následnou numerickou implementaci za předpokladu, že jádro tvoří dvoukomponentní směs. Model je popsán soustavou řídicích rovnic odvozených z konstitutivních vztahů a fyzikálních zákonů. Jsou odvozeny silná a slabá formulace, doplněné vhodnými okrajovými podmínkami, a celý systém je následně diskretizován v čase a prostoru. Model a jeho softwarová implementace jsou ověřeny na vhodně zvoleném referenčním problému. Nakonec je model aplikován na studium slapové deformace s cílem ukázat vliv vybraných poroelastických materiálových parametrů na slapovou sílu a deformaci v porovnání s čistě elastickým modelem.

Klíčová slova: slapová deformace, porozita, poroelasticita, počítačové modelování

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Introduction

The primary goal of this thesis is to assess the tidal deformation of Enceladus’s silicate, and potentially porous, core. Tidal deformation is highly sensitive to the moon’s internal structure. Enceladus’s internal structure is inferred through the mass and the moment-of-inertia factor $MOI = 0.33$ (Iess et al. [1] and McKinnon [5]), in combination with libration measurements (Thomas et al. [2] and Park et al. [4]). These data imply a relatively large and low-density core (see Figure 1). (Roberts [6]) proposed that such a core may consist of a non-consolidated rubble-pile, with pore spaces filled by water. The value of porosity is up to 30% for chondritic fraction of silicates in various degrees of serpentinisation (Choblet et al. [7]). This implies the core may be more deformable than previously thought, potentially acting as a driving force behind the observed geological activity (Roberts [6] and Choblet et al. [7]). The influence of a porous and deformable core on Enceladus’s tidal response has been investigated in previous studies, including Kamata [8], Rovira-Navarro et al. [9], and Liao et al. [10], which provide valuable context and motivation for the present work.

To explore the implications of this potentially porous and deformable core, this thesis develops a poroelastic model to quantify the tidal response of Enceladus’s interior. The first chapter focuses on the introduction of the poroelastic model and the derivation of the corresponding governing equations. The second chapter starts by endowing the system with proper boundary conditions, continues with

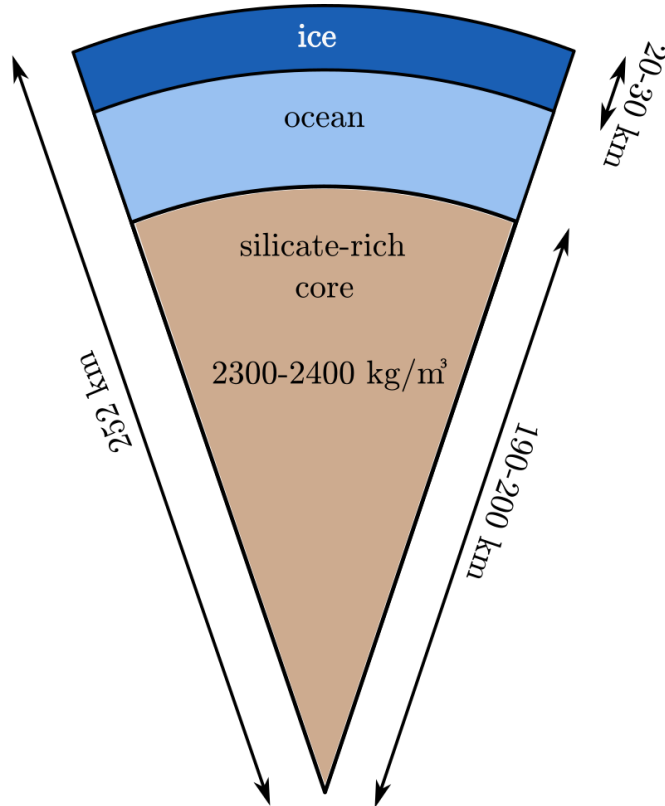


Figure 1 Possible structure of Enceladus. Values based on Iess et al. [1], Thomas et al. [2], Čadek et al. [3], and Park et al. [4].

the strong and weak formulations of the system, and ends with the monolithic form of the weak formulation. It is then discretised in the third chapter to allow for the numerical solvability of the problem. This model is then validated against the benchmark problem in the fourth chapter. The fifth and final chapter presents some selected results computed with various parameter values to demonstrate how they influence the pressure and displacement response of the core.

1 Governing Equations

This chapter serves as an introduction to the problem and provides an overview of the model. First, we describe the chosen poroelastic structure of the Enceladus core along with all its characteristics and properties. Following that, we derive the governing equations that fully describe our model. The first two equations are constitutive relations based on poroelastic theory, while the next three are derived from physical laws. We close the chapter with all five of them presented together to serve as an overview of our model.

The poroelastic model and its derivation are mainly based on the work of Alexander H.-D. Cheng (Cheng [11] and Detournay and Cheng [12]).

1.1 Problem Description

We do not simply consider just the porous core filled with fluid, we also want to factor in possible deformation of the solid part of the core, in other words, its elastic response. For that purpose, we take our material to be a two-component mixture with distinct phases (solid and fluid). From this point on, we shall refer to the material as *poroelastic*. For simplicity, we do not assume any viscoelastic or heat dependent behaviour. In this thesis, we focus solely on poroelasticity alone and leave these effects for future studies.

As stated above, we distinguish only between two distinct phases. The solid one is a silicate-rich material that we assume to be an *ideal porous medium* – homogeneous and isotropic solid. The fluid one is the same water as that in the subsurface ocean. For both materials, we prescribe their bulk moduli and material densities, and for the solid material also its shear modulus. The whole mixture then consists of a porous solid matrix with pores fully saturated with water. We consider this structure to be homogeneous at a scale one or two orders of magnitude larger than the grain size, and thus introduce the *volume surrounding the point of interest* – *Representative Elementary Volume* (REV, Bear [13]). This will soon come to use, as we now start to define the notation for our model. In particular, stresses and strains can be interpreted as quantities averaged over the REV.

The first and most intuitive one is the porosity, which represents the fraction of the REV occupied by pores (filled with fluid in our case). Normally, it is important to differentiate between areal (2D – the proportion of pores over an area) and volume (3D – the proportion of pores within a given volume). However, we assume that pores are randomly arranged without any specific pattern, and therefore the areal and volume porosity coincide, thanks to the *Delesse’s law* (Delesse [14]). We then take liberty and choose to define porosity as the volume porosity, that is, the fraction of fluid volume compared to the total volume of the REV. Denoting *total volume* as V and using the subscripts “ f ” and “ s ” for the *fluid and solid part of the volume*, respectively, the volume additivity

$$V = V_s + V_f$$

allows us to define *porosity* Φ as

$$\Phi = \frac{V_f}{V}.$$

Next, we examine the difference between so-called total and effective stresses and the role of pore pressure. In our case, the *total stress* $\boldsymbol{\sigma}$ is the overall stress resulting from all forces acting on REV averaged over its area. Since the REV consists of two components, a solid matrix and fluid-filled pores, the total stress can also be decomposed into two parts, each acting on a different phase – *solid partial stress* $\boldsymbol{\sigma}_s$ and *fluid partial stress* $\boldsymbol{\sigma}_f$

$$\boldsymbol{\sigma} = (1 - \Phi) \boldsymbol{\sigma}_s + \Phi \boldsymbol{\sigma}_f. \quad (1.1)$$

Given the inability of fluids to support shear stress, we are interested only in normal components of the fluid partial stress $\boldsymbol{\sigma}_f$. Further, assuming that the deviatoric part of the partial fluid stress averages to zero over the REV and taking the fluid as Newtonian (therefore not exhibiting any non-Newtonian phenomena such as non-zero normal stress differences), we introduce the *pore pressure* p as a scalar representation of the fluid part of the total stress (with value opposite to the mean normal stress due to different sign conventions for stress and pressure in relation to compression)

$$\boldsymbol{\sigma}_f = -p\mathbb{I}.$$

As its name suggests, the pore pressure represents the pressure of the fluid contained within the pores. In turn, the solid partial stress $\boldsymbol{\sigma}_s$ will be for the rest of this thesis referenced for brevity as an *effective stress*¹ as it is the component of total stress acting on the solid matrix and responsible for its (potential) deformation.

In addition to stresses, we also need to define strain quantities for both the solid and fluid components. For this purpose, we take the REV and consider it to be the *frame of the porous material* that surrounds the solid matrix and fluid-filled pores. We describe the frame deformation using the *displacement vector* $\vec{\mathbf{u}}$. We define the *total strain tensor* $\boldsymbol{\varepsilon}$ as the *small strain tensor*

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \vec{\mathbf{u}} + (\nabla \vec{\mathbf{u}})^T \right), \quad (1.2)$$

representing the deformation of the solid matrix of the porous material. The *volumetric change* e is then

$$e = \nabla \cdot \vec{\mathbf{u}}. \quad (1.3)$$

However, the volumetric strain of the fluid is not easily observable. The *total variation of the fluid content* ΔV_f consists of two parts

$$\Delta V_f = \Delta V_f^{(1)} + \Delta V_f^{(2)},$$

where $\Delta V_f^{(1)}$ is the component associated with interstitial fluid compression or expansion and $\Delta V_f^{(2)}$ is the component related to fluid exchange between the porous material sample and the external environment. For this reason, the *variation in fluid content* ζ , introduced by Biot (Biot [15]), is used as the fluid-related strain.

¹See Section 1.2 of Cheng [11] for discussion about the structure of the effective stress.

It is defined as the volume of fluid entering the solid frame per unit volume of the solid frame (Figure 1.1(Cheng [11])). Discerning solid and fluid displacement vectors (with respect to their own frames of references), with subscripts “ s ” and “ f ”, respectively, we can express ζ as

$$\zeta = \Phi \nabla \cdot (\vec{u}_f - \vec{u}_s).$$

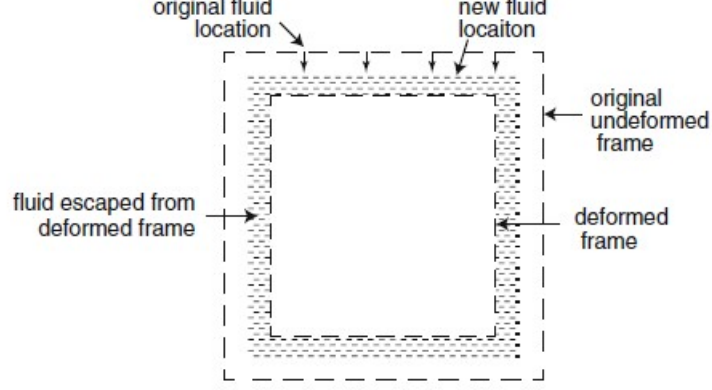


Figure 1.1 “Variation in fluid content: the shaded region indicates the volume of fluid escaped from the deformed frame as the negative variation in fluid content”

1.2 Constitutive equations

In this section, we introduce the first governing equations, the two constitutive equations for the linear isotropic poroelastic model based on linearised elasticity theory. Under the assumption of the reversibility of deformation in elastic materials, we then show that both equations contain only four independent material constants in total. We conclude the section with the final form of these first two governing equations.

1.2.1 Poroelastic Constitutive Equations

The constitutive equation of an isotropic linear elastic material, known also as *Hooke’s law* (Brđićka [16]), is

$$\sigma_\varepsilon = \lambda_u \mathbb{I}e + 2\mu \varepsilon, \quad (1.4)$$

where both material constants λ_u and μ are known as *Lamé parameters*. We use a subscript “ ε ” to indicate that σ_ε represents the stress of a purely elastic material. However, our poroelastic model consists of a fluid phase in addition to the solid one, and hence we need to modify the initial stress-strain relation. As discussed in the previous section, the inclusion of the fluid component introduces new quantities of stress (pore pressure p) and strain (variation in fluid content ζ). Therefore, we need two constitutive equations instead of one, as we now have two dynamic quantities (stresses) that we want to relate with kinematic ones.

The first equation stems from the above-mentioned Hooke’s law (1.4). For our case, we consider the total stress σ as the stress tensor instead of σ_ε as in

the original. We keep both terms on the right-hand side, but we also need to add a dependency on the strain of the fluid part, the variation in fluid content ζ . Applying linear theory gives us the *constitutive equation for stress*

$$\boldsymbol{\sigma} = \lambda_u \mathbb{I}e + 2\mu\boldsymbol{\varepsilon} - A\mathbb{I}\zeta, \quad (1.5)$$

where A is some new material constant. To ensure that all constants' values are positive, the last term's sign is negative, reflecting the opposite interpretation of strains for different phases.

The second equation also relates stress to strains, but this time the stress associated with the fluid phase, the pore pressure p . Once again, linear theory admits only one suitable form of the *constitutive equation for the pore pressure*

$$p = -\tilde{A}e + M\zeta, \quad (1.6)$$

where $\tilde{A}$ and M are another two new material constants.

1.2.2 Elimination of Constants

The two constitutive equations of our poroelastic model contain 5 material constants: the two Lamé parameters λ_u and μ , carried over from the purely elastic case, and the three new constants A , $\tilde{A}$ and M , coming from the incorporation of porosity and addition of the fluid phase to the model. We now show that only four of them are, in fact, independent.

The *work increment* dW caused by load on the system is (Biot [15])²

$$dW = \sigma_{ij} d\varepsilon_{ij} + p d\zeta.$$

Our material model is poroelastic – “poro” indicates the presence of pores in the structure, and “elastic” describes the behaviour. Therefore, given that elastic materials are characterised by their ability to return to their original state after deformation and by the fact that the *strain energy density* U is equal to the *mechanical work* W , it follows that

$$dU = dW.$$

Since the strain energy density U is a state function, the work increment dW must be an exact differential, and thus

$$\frac{\partial \sigma_{ij}}{\partial \zeta} = \frac{\partial p}{\partial \varepsilon_{ij}}. \quad (1.7)$$

Writing out the constitutive equations (1.5 and 1.6) in component form (using the *Einstein summation convention* and the *Kronecker delta* δ)

$$\begin{aligned} \sigma_{ij} &= \lambda_u \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} - A \delta_{ij} \zeta, \\ p &= -\tilde{A} \varepsilon_{kk} + M \zeta \end{aligned}$$

and taking their partial derivatives (1.7) yields

$$-A \delta_{ij} = -\tilde{A} \delta_{ij}.$$

²For detailed derivation see Chapter 4 of Cheng [11]

Hence, we identify $\tilde{A}$ with A to eliminate a material constant. We also choose to express A as a product of M and another material constant α

$$A = \alpha M.$$

With that, we have only four independent positive material constants: *Biot effective stress coefficient* α , *Biot Modulus* M and Lamé parameters λ_u and μ . Lastly, using the representation of the volumetric change e gives us the final form of the constitutive equations for the stress $\boldsymbol{\sigma}$ and the pore pressure p , respectively

$$\boldsymbol{\sigma} = \lambda_u \mathbb{I}e + 2\mu \boldsymbol{\varepsilon} - \alpha M \mathbb{I}\zeta, \quad (1.8)$$

$$p = -\alpha M e + M \zeta. \quad (1.9)$$

1.3 Physical Laws

In addition to the two constitutive equations (1.8 and 1.9) presented at the end of the previous section, we now derive, this time from physical laws, three more to obtain the full set of governing equations that describe our model. The equations and laws in question are The equilibrium equation from the balance of momentum, Darcy's law for fluid flow through pores, and The continuity equation from mass conservation.

1.3.1 Equilibrium Equation

The derivation process for the equilibrium equation is based on the *momentum balance equation*

$$\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \vec{b}, \quad (1.10)$$

where ρ stands for *mass density*, $\vec{v}$ is *velocity*, $\mathbf{T}$ denotes *Cauchy stress*, $\vec{b}$ represents *outer body forces* and $\frac{D}{Dt}$ is *material time derivative* is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla.$$

For our model, we need to separate balance equations for the fluid and solid phase. Each equation contains only quantities specific to that particular phase, with subscripts “*f*” and “*s*” indicating the proper association. However, we still need to quickly address two things.

The first is the density for the fluid and solid parts. Analogously to the concept of total stress, the *total density* of the poroelastic material ρ is the sum of the *material densities* ρ^m of the individual components weighted by their volume fractions: porosity and its complement for the fluid and solid parts, respectively. The total density ρ can be decomposed as

$$\rho = \rho_f + \rho_s = \Phi \rho_f^m + (1 - \Phi) \rho_s^m.$$

The second is the existence of interaction forces. When a material consists of two or more components, the general form of the momentum equation (1.10) has to be modified to account for the possibility that the components affect each

other. Thus, terms representing the *interaction forces* $\mathbf{I}_f$ and $\mathbf{I}_s$ must be added to the right-hand side of the relevant momentum equations for each component. Since we are dealing with only a two-part mixture, the principle of action and reaction allows us, without loss of generality, to identify $\mathbf{I}_s$ simply as $\mathbf{I}$ and to take $\mathbf{I}_f$ as its opposite.

With everything addressed, the individual balance equations are

$$\Phi \rho_f^m \frac{D\vec{v}_f}{Dt} = \nabla \cdot \Phi \boldsymbol{\sigma}_f - \Phi \rho_f^m \nabla B - \vec{\mathbf{I}}, \quad (1.11)$$

$$(1 - \Phi) \rho_s^m \frac{D\vec{v}_s}{Dt} = \nabla \cdot (1 - \Phi) \boldsymbol{\sigma}_s - (1 - \Phi) \rho_s^m \nabla B + \vec{\mathbf{I}}, \quad (1.12)$$

where B is the *tidal potential*. It is periodic, depends on position and is characterised long-wavelength degree two component.³

Summing up both equations, along with standard considerations of static equilibrium, the linearity of divergence operator, and using the splitting of total stress (1.1) results in the *equilibrium equation*

$$\nabla \cdot \boldsymbol{\sigma} + \rho \nabla B = 0. \quad (1.13)$$

1.3.2 Darcy's Law

Fluid transport within the interstitial space can be described by Darcy's law, a well-established empirical equation for seepage flow in non-deformable porous media. Assuming the presence of only deformations and negligible variation of fluid density, we can apply *Darcy's law* here without modification

$$\vec{q} = -\frac{\kappa}{\eta_f} (\nabla p - \rho_f^m \nabla B), \quad (1.14)$$

where $\vec{q}$ denotes the *Darcy flux*, η_f is the *dynamic viscosity of the fluid* and κ is the *intrinsic permeability*. Permeability is a physical quantity that represents the ability of the porous material to conduct fluid.⁴ For simplicity, in this thesis, we consider it to be a material constant related to our model.

1.3.3 Continuity Equation (Mass Conservation)

The continuity equation for the fluid is based on the traditional form of the *mass conservation equation* for a material with mass density ρ and velocity $\vec{v}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$

Applying mass conservation for a compressible fluid leads to the *continuity equation*

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \vec{q} = 0. \quad (1.15)$$

Note the linearised form of this equation, as the effects of fluid density variation have been neglected.

³The exact form used in code can be found in the fifth chapter.

⁴Permeability is related to various geometric properties of the porous material, such as pore or grain size, their structure, distribution or orientation, it is also related to the specific surface, interface density/aspect ratio of the pores and others. See Bercovici et al. [17], Cheng [11] for more detailed discussion.

1.4 Summary

To close this chapter, we present together all five governing equations we have derived throughout it that describe our poroelastic model.

The constitutive equation for stress (1.8)

$$\boldsymbol{\sigma} = \lambda_u \mathbb{I}(\nabla \cdot \vec{\mathbf{u}}) + 2\mu \boldsymbol{\varepsilon} - \alpha M \mathbb{I} \zeta. \quad (1.16)$$

The constitutive equation for pore pressure (1.9)

$$p = -\alpha M(\nabla \cdot \vec{\mathbf{u}}) + M\zeta. \quad (1.17)$$

The equilibrium equation (1.13)

$$\nabla \cdot \boldsymbol{\sigma} + \rho \nabla B = 0. \quad (1.18)$$

The Darcy's law (1.14)

$$\vec{\mathbf{q}} = -\frac{\kappa}{\eta_f} (\nabla p - \rho_f^m \nabla B). \quad (1.19)$$

The continuity equation (1.15)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \vec{\mathbf{q}} = 0. \quad (1.20)$$

2 Problem Formulation

Following the derivation of the governing equations at the end of the previous chapter, we combine them into two coupled partial differential equations. However, before that, we introduce the two boundary conditions, which together with the two PDEs form the complete strong formulation of our model. In what follows, we define the notion of the weak solution together with the requirements on the proper function spaces. Finally, we wrap up the chapter by combining our system of equations into one monolithic equation of displacement $\vec{u}$ and pore pressure p that we then solve numerically. However, first and foremost, let us introduce the core domain used throughout this chapter.

Let $\Omega \in \mathbb{R}^3$ denote the domain of the whole porous core, while $\Gamma \in \mathbb{R}^2$ represents its boundary (Figure 2.1).

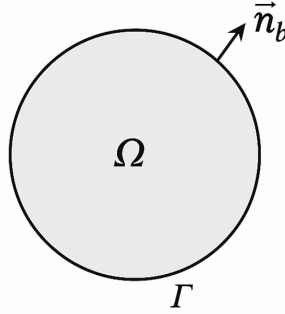


Figure 2.1 The core domain Ω with boundary Γ and unit outward normal $\vec{n}_b$

2.1 Boundary Conditions

Boundary conditions for our model come from mathematical expressions of two important physical properties of the core that we require. One is based on the permeability of this surface, thus allowing the ability between the porous core and the subsurface ocean of Enceladus to exchange fluid. The second stems from the assumption that the boundary of the core is a free surface.

2.1.1 Permeable Boundary

We want to allow¹ the option of fluid exchange between the core and the ocean. This requires a balance of force and continuity of pressure at the core-ocean interface. The ocean pressure at the core surface is balanced by the radial component of the stress tensor and also equals the pore pressure to preserve its continuity. This can be expressed by employing the *radial component of the stress tensor* σ_{rr} and the *ocean pressure* P_{ocean} ², as

$$\begin{aligned}\sigma_{rr} &= -P_{\text{ocean}} & \text{at } \Gamma, \\ p &= P_{\text{ocean}} & \text{at } \Gamma,\end{aligned}$$

¹In the opposite case we would prescribe zero radial Darcy flux at the boundary.

²For numerical implementation we use the relation $P_{\text{ocean}} = \rho_o B$, where B is tidal potential and ρ_o is the ocean density.

which further implies

$$\sigma_{rr} + p = 0 \quad \text{at } \Gamma;$$

another way of formulating our second boundary condition.

However, we are mostly interested in the part detailing equivalence of pressure values

$$p = P_{\text{ocean}} \quad \text{at } \Gamma. \quad (2.1)$$

When substituted in the constitutive equation for the pore pressure (1.17) we obtain the boundary condition in the form of

$$-\alpha M(\nabla \cdot \vec{u}) + M\zeta = P_{\text{ocean}} \quad \text{at } \Gamma,$$

which indicates that at the boundary ζ behaves similarly to $\nabla \cdot \vec{u}$.

2.1.2 Free Surface

The second condition describes the free surface property of the core boundary at the core-ocean interface when subject to tidal deformations. Its derivation is mostly analogous and closely follows the reasoning of Souček et al. [18] with the main difference being the opposite orientation of the vectors. In the reference source, it represents a bottom boundary condition for the ice shell at the shell-ocean interface, whereas in our case it serves as an upper boundary condition for the core-ocean interface. We describe the steps taken to derive this condition and finally present its form related to our model.

Due to tidal deformation, the material particles initially located on the surface of the core and described by the *position vectors* $\vec{r}_b$ move to new positions $\vec{r} = \vec{r}_b + \vec{u}$ and the *outward unit normal vector* changes to $\vec{n}_b + \delta\vec{n}_b$. After accounting for the effect of the hydrostatic pressure acting on the boundary, considering the deformation to be small ($\vec{n}_b + \delta\vec{n}_b \approx \vec{n}_b$), expanding the terms in Taylor series to first order in $\vec{u}$ and neglecting the second-order terms, the final form of the condition of free surface then reads

$$\sigma \cdot \vec{n}_b + u_r(\rho_o - \rho)g\vec{n}_b = -\rho_o B\vec{n}_b \quad \text{at } \Gamma, \quad (2.2)$$

where ρ_o denotes the *density of the ocean*, while ρ represents the *core density* and g stands for the *gravity acceleration*.

2.2 Equations System

With boundary conditions in place, we return to the governing equations presented at the end of the first chapter. Combining the five of them together results in two coupled PDEs for displacement $\vec{u}$ and pressure p .

The first is the *displacement equation* obtained by expressing the value of the stress tensor σ from the constitutive equation for stress (1.16) and substituting it into the equilibrium equation (1.18).

The second is the *pore pressure equation* describing the development of both the pressure and the divergence of displacement. Similarly to the displacement equation, it is achieved by substituting for the Darcy flux $\vec{q}$ in the continuity equation (1.20) from the Darcy's law (1.19).

The key part of the derivation process for each equation is the use of the constitutive equation for the pore pressure (1.17) to express the variation in fluid content ζ as a function of p and $\vec{u}$

$$\zeta = \frac{p + \alpha M(\nabla \cdot \vec{u})}{M}. \quad (2.3)$$

This couples pressure with displacement, and thus ensures that the system is formulated solely in terms of p and $\vec{u}$.

2.2.1 Displacement Equation

Since our model is poroelastic, we expect only an elastic response, and thus the partial differential equation for displacement does not contain any time derivatives, as we assume all changes of stress to be instant ($\frac{\partial \sigma}{\partial t} = 0$).

We start by identifying the *Lamé parameter* λ_u as (Cheng [11], B.2)

$$\lambda_u = K_u - \frac{2}{3}\mu, \quad (2.4)$$

where K_u is the *undrained bulk modulus* of the core, that describes the resistance to the compression of the solid porous material with pores filled by fluid in the case where the fluid is not allowed to escape. It combines the resistance of the solid porous structure and the entrapped fluid, which can be expressed as (Cheng [11], B.38)

$$K_u = K + \frac{K_f (K_s - K)^2}{K_f (K_s - K) + \Phi K_s (K_s - K_f)},$$

where K_s and K_f denote the *bulk moduli* of the solid and fluid part, respectively, and K is the *drained bulk moduli*, which measures the stiffness of the porous solid frame only, as it corresponds to the setting, when stress is applied to the porous material without any fluid inside. K is defined as (Cheng [11], B.39)

$$K = K_s (1 - \alpha),$$

and following inequality between various bulk moduli holds

$$K_s > K_u > K,$$

since K is the property of the solid frame alone, K_u also considers the effect of the fluid and K_s describes the solid material without holes or pores.

As a result of this, we can now use expressions for ϵ (1.2), ζ (2.3) and λ_u (2.4) to rewrite the constitutive equation for stress (1.16) as

$$\sigma = ((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}), \quad (2.5)$$

where $\mathbb{D}_d(\vec{u})$ denotes the *deviatoric part of the symmetric tensor of the displacement gradient*

$$\mathbb{D}_d(\vec{u}) = \frac{1}{2} \left(\nabla \vec{u} + (\nabla \vec{u})^T - \frac{2}{3}(\nabla \cdot \vec{u}) \mathbb{I} \right). \quad (2.6)$$

Finally, we can insert the value of σ from the equation derived above (2.5) into the equilibrium equation (1.18) to obtain the first of the two PDEs, the displacement equation, in the following form

$$\nabla \cdot \left[((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] + \rho \nabla B = 0. \quad (2.7)$$

We can also use the expression of $\boldsymbol{\sigma}$ (2.5) to rewrite the free surface boundary condition (2.2) as an equation for p and $\vec{\mathbf{u}}$

$$\left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{\mathbf{u}}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) \right] \cdot \vec{\mathbf{n}}_b + u_r(\rho_o - \rho)g\vec{\mathbf{n}}_b = -\rho_o B \vec{\mathbf{n}}_b \quad \text{at } \Gamma. \quad (2.8)$$

2.2.2 Pore Pressure Equation

The second PDE is an evolution equation for both pressure and divergence of displacement. Since Laplacian of the pressure is also present, the final equation can thus be seen as a modified pressure diffusion equation, hence further highlighting the coupling between pore pressure and displacement in a fluid-filled porous material.

Consider the continuity equation (1.20), then, taking the value of $\vec{\mathbf{q}}$ from the Darcy's law (1.19) and once again making use of the representation of ζ (2.3) gives us the pore pressure equation

$$\frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial(\nabla \cdot \vec{\mathbf{u}})}{\partial t} - \frac{\kappa}{\eta_f} \Delta p + \frac{\kappa}{\eta_f} \rho_f^m \Delta B = 0. \quad (2.9)$$

2.3 Strong Formulation

With all the boundary conditions and PDEs to fully describe our model formulated, we present the whole system, for p and $\vec{\mathbf{u}}$, here in a clean and concise way before defining the notion of its weak form.

Displacement equation (2.7)

$$\nabla \cdot \left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{\mathbf{u}}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) \right] + \rho \nabla B = 0 \quad \text{in } \Omega. \quad (2.10)$$

Pore pressure equation (2.9)

$$\frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial(\nabla \cdot \vec{\mathbf{u}})}{\partial t} - \frac{\kappa}{\eta_f} \Delta p + \frac{\kappa}{\eta_f} \rho_f^m \Delta B = 0 \quad \text{in } \Omega. \quad (2.11)$$

Free surface boundary condition (2.8)

$$\left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{\mathbf{u}}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) \right] \cdot \vec{\mathbf{n}}_b + u_r(\rho_o - \rho)g\vec{\mathbf{n}}_b = -\rho_o B \vec{\mathbf{n}}_b \quad \text{at } \Gamma. \quad (2.12)$$

Permeable boundary condition (2.1)

$$p = P_{\text{ocean}} \quad \text{at } \Gamma. \quad (2.13)$$

2.4 Weak Formulation

With the strong formulation at hand, we now proceed to define the notion of a weak solution for the system.

Initially, we derive the weak formulation for each of the two partial differential equations (2.10) and (2.11) individually; assuming that either p or $\vec{\mathbf{u}}$ is known. This follows the standard procedure – multiply the equation by a suitable test

function, integrate over core domain, use integration by parts, and apply boundary conditions to the resulting surface integrals.

In the end, we combine everything into one monolithic equation for the pore pressure p and the displacement $\vec{\mathbf{u}}$, corresponding to the governing equations and satisfying all the boundary conditions. The steps taken to obtain a numerical solution to this equation are then detailed in the next chapter.

However, before starting with anything described above, we need to ensure the uniqueness of the weak solution, and to this end, we impose two additional conditions that guarantee zero displacement of the centre of figure

$$\int_{\Omega} \vec{\mathbf{u}} \, d\Omega = 0, \quad (2.14)$$

and zero net-body rotation

$$\int_{\Omega} \vec{\mathbf{r}} \times \vec{\mathbf{u}} \, d\Omega = 0, \quad (2.15)$$

where $\vec{\mathbf{r}}$ denotes the *position vector from the centre of the sphere*.

Also, since establishing the concept of the weak solution and proving its existence in a detailed and rigorous way is not the focus of this thesis, we assume that the pore pressure p and divergence of the displacement $(\nabla \cdot \vec{\mathbf{u}})$ are of sufficient time regularity and the space $H^1(\Omega; \mathbb{R}^3)$ simply conforms to our chosen formulation.

2.4.1 Displacement Equation

To derive the notion of a weak solution, assuming that the pore pressure p is known, we first multiply both sides of the displacement equation (2.10) by the vector-valued test function $\vec{\mathbf{w}} \in H^1(\Omega; \mathbb{R}^3)$, then integrate over Ω and use integration by parts to obtain

$$\begin{aligned} & - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}) (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega + \int_{\Omega} \alpha p (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) : \mathbb{D}_d(\vec{\mathbf{w}}) \, d\Omega \\ & + \int_{\Gamma} \left[((K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}) - \alpha p) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) \right] \cdot \vec{\mathbf{n}}_b \cdot \vec{\mathbf{w}} \, d\Gamma = - \int_{\Omega} \rho \nabla B \cdot \vec{\mathbf{w}} \, d\Omega. \end{aligned}$$

Utilising the free surface boundary condition (2.12) in the surface integral and implementing boundary conditions (2.14 and 2.15), which guarantee the uniqueness of solution, using Lagrange multipliers $\vec{\mathbf{h}}, \vec{\mathbf{j}} \in \mathbb{R}^3$ leads to

$$\begin{aligned} & - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}) (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega + \int_{\Omega} \alpha p (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega \\ & - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{\mathbf{u}}) : \mathbb{D}_d(\vec{\mathbf{w}}) \, d\Omega - \int_{\Gamma} ((\rho_o - \rho) g (\vec{\mathbf{u}} \cdot \vec{\mathbf{e}}_r) + \rho_o B) (\vec{\mathbf{w}} \cdot \vec{\mathbf{n}}_b) \, d\Gamma \\ & + \int_{\Omega} \vec{\mathbf{h}} \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} \vec{\mathbf{s}} \cdot \vec{\mathbf{u}} \, d\Omega + \int_{\Omega} (\vec{\mathbf{j}} \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} (\vec{\mathbf{u}} \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{t}} \, d\Omega \\ & = - \int_{\Omega} \rho \nabla B \cdot \vec{\mathbf{w}} \, d\Omega, \end{aligned} \quad (2.16)$$

where for all $(\vec{\mathbf{w}}, \vec{\mathbf{s}}, \vec{\mathbf{t}}) \in H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ we seek the triplet $(\vec{\mathbf{u}}, \vec{\mathbf{h}}, \vec{\mathbf{j}}) \in H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $\vec{\mathbf{u}}$ satisfies for known pore pressure p the free surface boundary condition (2.12) and equation (2.16) holds.

2.4.2 Pore Pressure Equation

To derive the notion of a weak solution for the pore pressure equation (2.11), we proceed analogously to the previous case of the displacement equation. Multiplying both sides of the equation by a scalar-valued test function $w \in H^1(\Omega; \mathbb{R})$, integrating over the whole domain Ω and using integration by parts, all under the assumption that displacement $\vec{u}$ is known, results in

$$\begin{aligned} \int_{\Omega} \frac{1}{M} \frac{\partial p}{\partial t} w \, d\Omega + \int_{\Omega} \alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t} w \, d\Omega + \int_{\Omega} \frac{\kappa}{\eta_f} (\nabla p) \cdot (\nabla w) \, d\Omega - \int_{\Gamma} \frac{\kappa}{\eta_f} \frac{\partial p}{\partial \vec{n}_b} w \, d\Gamma \\ = - \int_{\Omega} \frac{\kappa}{\eta_f} \rho_f^m w \Delta B \, d\Omega. \end{aligned} \quad (2.17)$$

Although we have the boundary condition for the pore pressure p (2.13), we cannot use it in the surface integral, as it contains the normal derivative of the pressure about which the Dirichlet-type condition does not provide any information. Despite all this, we will still manage to leverage information about the behaviour of the pore pressure at the surface because of Nitsche's method. This will be shown in detail in the following chapter during the spatial discretisation process.

With that in mind, for any $w \in H^1(\Omega; \mathbb{R})$ and known displacement $\vec{u}$ we are looking for $p \in H^1(\Omega; \mathbb{R})$ such that equation (2.16) holds and both, the free surface (2.12) and the permeable boundary (2.13), boundary conditions are satisfied.

2.4.3 Weak System

Before moving on to the final result of this chapter, the monolithic equation, we first write down the whole weak system in a clear and concise way.

Displacement equation (2.16)

$$\begin{aligned} - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{u}) (\nabla \cdot \vec{w}) \, d\Omega + \int_{\Omega} \alpha p (\nabla \cdot \vec{w}) \, d\Omega \\ - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{u}) : \mathbb{D}_d(\vec{w}) \, d\Omega - \int_{\Gamma} ((\rho_o - \rho)g(\vec{u} \cdot \vec{e}_r) + \rho_o B) (\vec{w} \cdot \vec{n}_b) \, d\Gamma \\ + \int_{\Omega} \vec{h} \cdot \vec{w} \, d\Omega + \int_{\Omega} \vec{s} \cdot \vec{u} \, d\Omega + \int_{\Omega} (\vec{j} \times \vec{r}) \cdot \vec{w} \, d\Omega + \int_{\Omega} (\vec{u} \times \vec{r}) \cdot \vec{t} \, d\Omega \\ = - \int_{\Omega} \rho \nabla B \cdot \vec{w} \, d\Omega. \end{aligned} \quad (2.18)$$

Pore pressure equation (2.17)

$$\begin{aligned} \int_{\Omega} \frac{1}{M} \frac{\partial p}{\partial t} w \, d\Omega + \int_{\Omega} \alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t} w \, d\Omega + \int_{\Omega} \frac{\kappa}{\eta_f} (\nabla p) \cdot (\nabla w) \, d\Omega - \int_{\Gamma} \frac{\kappa}{\eta_f} \frac{\partial p}{\partial \vec{n}_b} w \, d\Gamma \\ = - \int_{\Omega} \frac{\kappa}{\eta_f} \rho_f^m w \Delta B \, d\Omega. \end{aligned} \quad (2.19)$$

Free surface boundary condition (2.12)

$$\left[((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] \cdot \vec{n}_b + u_r (\rho_o - \rho) g \vec{n}_b = -\rho_o B \vec{n}_b \quad \text{at } \Gamma. \quad (2.20)$$

Permeable boundary boundary condition (2.13)

$$p = P_{\text{ocean}} \quad \text{at } \Gamma. \quad (2.21)$$

We seek $(p, \vec{u}, \vec{h}, \vec{j}) \in H^1(\Omega; \mathbb{R}) \times H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ such that for all $(w, \vec{w}, \vec{s}, \vec{t}) \in H^1(\Omega; \mathbb{R}) \times H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ equations (2.18) and (2.19) hold and the boundary conditions (2.20) and (2.21) are satisfied.

2.4.4 Monolithic Formulation

The weak system (2.18 - 2.19) contains two PDEs with coupled variables – pore pressure p and displacement $\vec{u}$. For the purpose of numerical solvability, we elect to sum these partial differential equations into one monolithic equation of p and $\vec{u}$. After performing the necessary time and spatial discretisation in the next chapter, it will be implemented in code together with both boundary conditions to obtain a numerical solution for our model.

The sum of the displacement equation (2.18) and the pore pressure equation (2.19) results in the following *monolithic equation*:

$$\begin{aligned} \int_{\Omega} \frac{1}{M} \frac{\partial p}{\partial t} w \, d\Omega + \int_{\Omega} \alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t} w \, d\Omega + \int_{\Omega} \frac{\kappa}{\eta_f} (\nabla p) \cdot (\nabla w) \, d\Omega - \int_{\Gamma} \frac{\kappa}{\eta_f} \frac{\partial p}{\partial \vec{n}_b} w \, d\Gamma \\ - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{u}) (\nabla \cdot \vec{w}) \, d\Omega + \int_{\Omega} \alpha p (\nabla \cdot \vec{w}) \, d\Omega \\ - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{u}) : \mathbb{D}_d(\vec{w}) \, d\Omega - \int_{\Gamma} ((\rho_o - \rho)g(\vec{u} \cdot \vec{e}_r) + \rho_o B) (\vec{w} \cdot \vec{n}_b) \, d\Gamma \\ + \int_{\Omega} \vec{h} \cdot \vec{w} \, d\Omega + \int_{\Omega} \vec{s} \cdot \vec{u} \, d\Omega + \int_{\Omega} (\vec{j} \times \vec{r}) \cdot \vec{w} \, d\Omega + \int_{\Omega} (\vec{u} \times \vec{r}) \cdot \vec{t} \, d\Omega \\ = - \int_{\Omega} \frac{\kappa}{\eta_f} \rho_f^m w \Delta B \, d\Omega - \int_{\Omega} \rho \nabla B \cdot \vec{w} \, d\Omega. \end{aligned} \quad (2.22)$$

Together with free surface boundary condition (2.20)

$$\left[((K_u - \alpha^2 M) (\nabla \cdot \vec{u}) - \alpha p) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] \cdot \vec{n}_b + u_r (\rho_o - \rho) g \vec{n}_b = -\rho_o B \vec{n}_b \quad \text{at } \Gamma \quad (2.23)$$

and permeable boundary boundary condition (2.21)

$$p = P_{\text{ocean}} \quad \text{at } \Gamma, \quad (2.24)$$

the final weak formulation then reads:

Find a quartet $(p, \vec{u}, \vec{h}, \vec{j}) \in H^1(\Omega; \mathbb{R}) \times H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ such that for all $(w, \vec{w}, \vec{s}, \vec{t}) \in H^1(\Omega; \mathbb{R}) \times H^1(\Omega; \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$ the monolithic equation (2.22) holds and the boundary conditions (2.23) and (2.24) are satisfied.

3 Numerical Solution

This chapter details steps needed to facilitate the numerical solution of the monolithic form of the weak system, the formulation of which was the key and final result of the previous chapter.

We seek the ability to solve the system numerically by employing the finite element method. However, before proceeding to spatial discretisation and the desired formulation, we first need to deal with the time derivatives present in the monolithic equation (2.22). This is achieved with the first-order backward Euler method.

In what follows, the Galerkin method and the Nitsche method yield our goal and primary result of this chapter, the weak system discretised in time and space, formulated in the context of FEM. This system is then implemented in code and solved using the FEniCS software package (Alnaes et al. [19]).

We modify the baseline code based on Souček et al. [18] starting from an elastic model for the ice shell of Enceladus. The work carried out as part of this thesis includes the implementation of the full poroelastic model, different integration domain, boundary conditions, and the Nitsche method, and the addition of new constants and variables.

3.1 Time Discretisation

We approximate the time derivatives of the pressure and the divergence of the displacement, using one-step implicit time schemes with constant *time step* Δt

$$\frac{\partial p}{\partial t} \approx \frac{p^k - p^{k-1}}{\Delta t}; \quad (3.1)$$

$$\frac{\partial(\nabla \cdot \vec{\mathbf{u}})}{\partial t} \approx \frac{(\nabla \cdot \vec{\mathbf{u}}^k) - (\nabla \cdot \vec{\mathbf{u}}^{k-1})}{\Delta t}; \quad (3.2)$$

where the superscript “ k ” denotes the value at time $t_k = t_0 + k\Delta t$.

We take the monolithic system, the monolithic equation (2.22) and the boundary conditions (2.23) and (2.24), and apply the above-mentioned approximations (3.1) and (3.2), we also consider all variables to be evaluated at the k -th timestep; this results in the following set of equations:

Time-discretised monolithic equation

$$\begin{aligned} & \int_{\Omega} \frac{1}{M\Delta t} p^k w \, d\Omega - \int_{\Omega} \frac{1}{M\Delta t} p^{k-1} w \, d\Omega + \int_{\Omega} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}^k) w \, d\Omega \\ & - \int_{\Omega} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}^{k-1}) w \, d\Omega + \int_{\Omega} \frac{\kappa}{\eta_f} (\nabla p^k) \cdot (\nabla w) \, d\Omega - \int_{\Gamma} \frac{\kappa}{\eta_f} \frac{\partial p^k}{\partial \vec{\mathbf{n}}_b} w \, d\Gamma \\ & - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}^k) (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega + \int_{\Omega} \alpha p^k (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega \\ & - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{\mathbf{u}}^k) : \mathbb{D}_d(\vec{\mathbf{w}}) \, d\Omega - \int_{\Gamma} ((\rho_o - \rho)g(\vec{\mathbf{u}}^k \cdot \vec{\mathbf{e}}_r) + \rho_o B^k) (\vec{\mathbf{w}} \cdot \vec{\mathbf{n}}_b) \, d\Gamma \\ & + \int_{\Omega} \vec{\mathbf{h}}^k \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} \vec{\mathbf{s}} \cdot \vec{\mathbf{u}}^k \, d\Omega + \int_{\Omega} (\vec{\mathbf{j}}^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} (\vec{\mathbf{u}}^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{t}} \, d\Omega \\ & = - \int_{\Omega} \frac{\kappa}{\eta_f} \rho_f^m w \Delta B^k \, d\Omega - \int_{\Omega} \rho \nabla B^k \cdot \vec{\mathbf{w}} \, d\Omega. \end{aligned} \quad (3.3)$$

Time-discretised free surface boundary condition at Γ

$$\left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{u}^k) - \alpha p^k \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}^k) \right] \cdot \vec{n}_b + u_r^k(\rho_o - \rho)g\vec{n}_b = -\rho_o B^k \vec{n}_b. \quad (3.4)$$

Time-discretised permeable boundary condition

$$p^k = P_{\text{ocean}}^k \quad \text{at } \Gamma. \quad (3.5)$$

3.2 Spatial Discretisation

With a time-discretised system at hand, there is only one step left before space discretisation. First, we need to deal with the permeable boundary condition (3.5) and its effect on the monolithic equation (3.3). As mentioned in the previous section, because of the presence of a normal derivative of the pressure in the boundary integral, we cannot simply substitute the boundary value in the equation. However, we can impose the condition weakly thanks to the Nitsche method.

The following formulation is based on text from Blechta et al. [20]. The notation in use

$\mathcal{B}(\cdot, \cdot)$	<i>bilinear form,</i>
$\mathcal{F}(\cdot)$	<i>linear functional,</i>
V_h	<i>finite-dimensional function space,</i>
$\langle \cdot, \cdot \rangle$	<i>duality pairing,</i>
E	<i>edge of a finite element,</i>
h_E	<i>length of edge E,</i>
β	<i>penalty parameter (positive).¹</i>

Using the above, for the problem

$$\text{Find } u_h \in V_h \text{ such that: } \mathcal{B}(u_h, v) = \mathcal{F}(v) \quad \forall v \in V_H$$

with boundary condition

$$u = u_0 \quad \text{at } \Gamma$$

where the bilinear form contains the term

$$- \langle \nabla u \cdot \mathbf{n}, v \rangle_\Gamma$$

the antisymmetric Nitsche method is of the form

$$\begin{aligned} \mathcal{B}(u, v) + \langle \nabla v \cdot \mathbf{n}, u \rangle_\Gamma + \beta \sum_{E \in \Gamma} h_E^{-1} \langle u, v \rangle_E &= \mathcal{F}(v) + \langle u_0, \nabla v \cdot \mathbf{n} \rangle_\Gamma \\ &+ \beta \sum_{E \in \Gamma} h_E^{-1} \langle u_0, v \rangle_E. \end{aligned} \quad (3.6)$$

Therefore, implementing the Nitsche method for our case, we identify u as pressure p and v as the scalar test function w , and we add

$$\int_\Gamma \frac{\kappa}{\eta_f} p^k \frac{\partial w}{\partial \vec{n}_b} d\Gamma - \int_\Gamma \frac{\kappa}{\eta_f} P_{\text{ocean}}^k \frac{\partial w}{\partial \vec{n}_b} d\Gamma$$

¹We proceed with the usual choice $\beta = 1$.

to the left-hand side, and

$$-\beta \sum_{E \in \Gamma} h_E^{-1} \int_E (p^k - P_{\text{ocean}}^k) w \, dE$$

to the right-hand side.

Thus, the time-discrete monolithic equation (3.3) becomes

$$\begin{aligned} & \int_{\Omega} \frac{1}{M \Delta t} p^k w \, d\Omega - \int_{\Omega} \frac{1}{M \Delta t} p^{k-1} w \, d\Omega + \int_{\Omega} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}^k) w \, d\Omega \\ & - \int_{\Omega} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}^{k-1}) w \, d\Omega + \int_{\Omega} \frac{\kappa}{\eta_f} (\nabla p^k) \cdot (\nabla w) \, d\Omega - \int_{\Gamma} \frac{\kappa}{\eta_f} \frac{\partial p^k}{\partial \vec{\mathbf{n}}_b} w \, d\Gamma \\ & - \int_{\Omega} (K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}^k) (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega + \int_{\Omega} \alpha p^k (\nabla \cdot \vec{\mathbf{w}}) \, d\Omega \\ & - \int_{\Omega} 2\mu \mathbb{D}_d(\vec{\mathbf{u}}^k) : \mathbb{D}_d(\vec{\mathbf{w}}) \, d\Omega - \int_{\Gamma} ((\rho_o - \rho)g(\vec{\mathbf{u}}^k \cdot \vec{\mathbf{e}}_r) + \rho_o B^k) (\vec{\mathbf{w}} \cdot \vec{\mathbf{n}}_b) \, d\Gamma \\ & + \int_{\Omega} \vec{\mathbf{h}}^k \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} \vec{\mathbf{s}} \cdot \vec{\mathbf{u}}^k \, d\Omega + \int_{\Omega} (\vec{\mathbf{j}}^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{w}} \, d\Omega + \int_{\Omega} (\vec{\mathbf{u}}^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{t}} \, d\Omega \\ & + \int_{\Gamma} \frac{\kappa}{\eta_f} p^k \frac{\partial w}{\partial \vec{\mathbf{n}}_b} \, d\Gamma - \int_{\Gamma} \frac{\kappa}{\eta_f} P_{\text{ocean}}^k \frac{\partial w}{\partial \vec{\mathbf{n}}_b} \, d\Gamma = -\beta \sum_{E \in \Gamma} h_E^{-1} \int_E (p^k - P_{\text{ocean}}^k) w \, dE \\ & - \int_{\Omega} \frac{\kappa}{\eta_f} \rho_f^m w \Delta B^k \, d\Omega - \int_{\Omega} \rho \nabla B^k \cdot \vec{\mathbf{w}} \, d\Omega. \end{aligned} \tag{3.7}$$

With the Nitsche method successfully applied, all that remains is to discretise the system consisting of equation (3.7) and boundary conditions (3.4) and (3.5). For this purpose, we use the Galerkin method. We replace the function spaces H^1 with P_1 for the pore pressure p and P_2 for the displacement $\vec{\mathbf{u}}$, respectively, where P_n is the *finite element space of continuous piecewise polynomial functions of degree $\leq n$* . The difference in the choice of space for p and $\vec{\mathbf{u}}$ is motivated by the desire for a consistent discrete representation of the variation in fluid content ζ , since it depends on p and $(\nabla \cdot \vec{\mathbf{v}})$ (2.3), thus the higher order of discretisation for displacement $\vec{\mathbf{u}}$. We triangulate the domain Ω using tetrahedrons as our finite elements. In what follows, finite-dimensional approximations of quantities are indicated with the subscript “ h ”. The discretised equations are given below.

Space-time discretised monolithic equation

$$\begin{aligned}
& \int_{\Omega_h} \frac{1}{M\Delta t} p_h^k w_h \, d\Omega_h - \int_{\Omega_h} \frac{1}{M\Delta t} p_h^{k-1} w_h \, d\Omega_h + \int_{\Omega_h} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}_h^k) w_h \, d\Omega_h \\
& - \int_{\Omega_h} \frac{\alpha}{\Delta t} (\nabla \cdot \vec{\mathbf{u}}_h^{k-1}) w_h \, d\Omega_h + \int_{\Omega_h} \frac{\kappa}{\eta_f} (\nabla p_h^k) \cdot (\nabla w_h) \, d\Omega_h - \int_{\Gamma_h} \frac{\kappa}{\eta_f} \frac{\partial p_h^k}{\partial \vec{\mathbf{n}}_b} w_h \, d\Gamma_h \\
& - \int_{\Omega_h} (K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}_h^k) (\nabla \cdot \vec{\mathbf{w}}_h) \, d\Omega_h + \int_{\Omega_h} \alpha p_h^k (\nabla \cdot \vec{\mathbf{w}}_h) \, d\Omega_h \\
& - \int_{\Omega_h} 2\mu \mathbb{D}_d(\vec{\mathbf{u}}_h^k) : \mathbb{D}_d(\vec{\mathbf{w}}_h) \, d\Omega_h \\
& - \int_{\Gamma_h} ((\rho_o - \rho) g (\vec{\mathbf{u}}_h^k \cdot \vec{\mathbf{e}}_r) + \rho_o B^k) (\vec{\mathbf{w}}_h \cdot \vec{\mathbf{n}}_b) \, d\Gamma_h \\
& + \int_{\Omega_h} \vec{\mathbf{h}}^k \cdot \vec{\mathbf{w}}_h \, d\Omega_h + \int_{\Omega_h} \vec{\mathbf{s}} \cdot \vec{\mathbf{u}}_h^k \, d\Omega_h + \int_{\Omega_h} (\vec{\mathbf{j}}^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{w}}_h \, d\Omega_h \\
& + \int_{\Omega_h} (\vec{\mathbf{u}}_h^k \times \vec{\mathbf{r}}) \cdot \vec{\mathbf{t}} \, d\Omega_h + \int_{\Gamma_h} \frac{\kappa}{\eta_f} p_h^k \frac{\partial w_h}{\partial \vec{\mathbf{n}}_b} \, d\Gamma_h - \int_{\Gamma_h} \frac{\kappa}{\eta_f} P_{\text{ocean}}^k \frac{\partial w_h}{\partial \vec{\mathbf{n}}_b} \, d\Gamma_h \\
& = -\beta \sum_{E \in \Gamma_h} h_E^{-1} \int_E (p_h^k - P_{\text{ocean}}^k) w_h \, dE \\
& - \int_{\Omega_h} \rho \nabla B^k \cdot \vec{\mathbf{w}}_h \, d\Omega_h - \int_{\Omega_h} \frac{\kappa}{\eta_f} \rho_f^m w_h \Delta B^k \, d\Omega_h.
\end{aligned} \tag{3.8}$$

Space-time discretised free surface boundary condition at Γ

$$\left[((K_u - \alpha^2 M) (\nabla \cdot \vec{\mathbf{u}}_h^k) - \alpha p_h^k) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{\mathbf{u}}_h^k) \right] \cdot \vec{\mathbf{n}}_b + (u_r)_h^k (\rho_o - \rho) g \vec{\mathbf{n}}_b = -\rho_o B^k \vec{\mathbf{n}}_b. \tag{3.9}$$

Space-time discretised permeable boundary condition

$$p_h^k = P_{\text{ocean}}^k \quad \text{at } \Gamma. \tag{3.10}$$

With all that in mind, the formulation in the context of finite elements reads:

Find a quartet $(p_h^k, \vec{\mathbf{u}}_h^k, \vec{\mathbf{h}}^k, \vec{\mathbf{j}}^k) \in P_1 \times (P_2)^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ such that for all $(w_h, \vec{\mathbf{w}}_h, \vec{\mathbf{s}}, \vec{\mathbf{t}}) \in P_1 \times (P_2)^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, the equations (3.8), (3.9) and (3.10) hold.

4 Model Validation

This chapter details the steps to verify that the model derived by us and its code implementation are done correctly and produce expected results compared to the benchmark case. First, we describe the chosen benchmark – *Cryer problem*. Then, we derive its formulation from our governing equations to justify the choice of reference problem. Finally, using material constants corresponding as closely as possible to the original setting of the benchmark problem, we compare our computed numerical solution to the analytical one and show that our code manages to properly replicate the Mandel-Cryer effect present in the Cryer problem.

4.1 Cryer Problem

The Cryer problem is a foundational benchmark in poroelasticity, originally formulated in 1963 (Cryer [21]), with a solution for the pore pressure and radial displacement derived only in the centre and at the surface, respectively. The full solution was later derived in 1991 (Mason et al. [22]).

It is a spherically symmetric problem that examines the time-dependent behaviour of an initially undisturbed saturated porous sphere with permeable boundary subject to a suddenly applied constant radial compressive stress. Despite the permeable boundary allowing the fluid to escape freely, rapid loading results in an undrained response in the form of a uniform pore pressure rise at the instant of load application; this is known as the *Skempton pore pressure effect* (Skempton [23]). The pore pressure then starts to dissipate at the edges of the domain as the fluid begins to drain through the permeable boundary. Due to the coupling between the pressure and the elastic deformation, the specimen becomes inhomogeneous as the outer layer of the sphere effectively softens and the inner part stiffens instead. This results in an increase of total compressive stress within the sphere generating higher pore pressure which can rise well above¹ its initial value. This non-monotonic pressure behaviour can be observed at small times. The phenomenon is known as the *Mandel-Cryer effect* (Schiffman et al. [24]). When validating numerical solution to our model against the Cryer problem as a benchmark, the exhibition of this effect will be sought after property.

In this section, we first present the Cryer problem formulation together with boundary conditions and then briefly comment on how to obtain the analytical solution.

4.1.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^3$ be a saturated porous sphere of radius r_0 . The spherical symmetry of the Cryer problem means that all variables are functions of the *radial distance* r and the *time* t only. Hence, for displacement $\vec{u}$ only the radial component u_r can exist

$$\vec{u} = u_r \vec{e}_r = u_r(r, t) \vec{e}_r, \quad (4.1)$$

¹“Cryer’s analysis showed that during the early stages of consolidation the pore water pressure at the centre of the sphere could rise almost 60 % above the initial pressure, before it starts to decline.” (Cheng [11])

and similarly for stresses and strains, only the normal components are non-zero.

Cryer problem is described by two governing equations. The first of them is the Navier equation

$$\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial r^2 u_r}{\partial r} \right) = \frac{\eta}{\mu} \frac{\partial p}{\partial r}, \quad (4.2)$$

with η being the *poroelastic stress coefficient*. The second is then the pressure diffusion equation

$$\frac{\partial p}{\partial t} - c \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial p}{\partial r} \right) = -\frac{3\alpha}{S} \frac{dA_1}{dt}, \quad (4.3)$$

where c stands for the *consolidation coefficient*, S denotes the *storage coefficient* and A_1 is an integration constant obtainable after integrating the Navier equation (4.2) once.

To properly close the system, these equations are accompanied by the corresponding boundary conditions. The specimen is initially undisturbed with radial stress applied at $t = 0^+$ which together with the drained condition (zero pressure at the surface) gives us the conditions at the boundary of the sphere

$$\sigma_{rr} = -P_0 H(t - 0); \quad p = 0 \quad \text{at } r = r_0, \quad (4.4)$$

with r_0 denoting the *sphere radius*, $H(\cdot)$ representing the *Heaviside function* and P_0 being the *loading pressure*.

Furthermore, the displacement and the radial change of the pressure in the centre of the sphere need to be zero

$$u_r = 0; \quad \frac{\partial p}{\partial r} = 0 \quad \text{at } r = 0. \quad (4.5)$$

4.1.2 Analytical Solution

The derivation of the analytical solution to the Cryer problem is quite technical, and since it is not the main focus of this thesis, we present only the essential steps, while full details can be found in Section 7.10 of Cheng [11].

Derivation of a solution in the spectral domain:

1. Laplace transform of the pressure diffusion equation (4.3) results in the second-order ordinary differential equation (ODE).
2. Solve the ODE and use the boundedness of the solution and the boundary condition to identify the integration constants.
3. Repeat the previous steps for the Navier equation (4.2) to obtain a solution for the normal radial component of the stress.
4. Use the boundary condition (4.4) identifying σ_{rr} and solve for the unknown arbitrary function $\hat{A}_1$ (Laplace transform of the integration constant A_1 from the pressure diffusion equation (4.3)).
5. Substitute the above into the result of the second step to achieve the finalised form of the solution for pressure p .

6. The solutions for the normal stress components and the displacement can now be obtained analogously.

The last step is the inverse Laplace transform of the solution in the spectral domain for the pressure, which yields the desired solution in the time domain.

The analytical solution for the normalised pressure is then

$$\frac{p(r, t)}{P_0} = \sum_{n=1}^{\infty} \frac{18 (\nu_u - \nu)^2}{\eta E(x_n)} \left[\frac{\sin(r^* \sqrt{x_n})}{r^* \sin \sqrt{x_n}} - 1 \right] e^{-x_n t^*}, \quad (4.6)$$

where

$$E(x_n) = (1 - \nu)^2 (1 + \nu_u)^2 x_n - 18 (1 + \nu) (\nu_u - \nu) (1 - \nu_u) \quad (4.7)$$

and x_n are the positive roots of the algebraic equation

$$\tan \sqrt{x_n} = \frac{6 (\nu_u - \nu) \sqrt{x_n}}{6 (\nu_u - \nu) - (1 - \nu) (1 + \nu_u) x_n} : \quad x_n > 0. \quad (4.8)$$

The newly introduced parameters are the *undrained Poisson ratio* ν_u and two dimensionless ones

$$r^* = \frac{r}{r_0}, \quad (4.9)$$

$$t^* = \frac{ct}{r_0^2}, \quad (4.10)$$

radius and time, respectively. The characteristic time $t^* = 1$ corresponds to the *poroelastic diffusion time* t_d which is of the following form

$$t_d = \frac{r_0^2}{c}. \quad (4.11)$$

4.2 Application to Our Model

First, we need to show that the strong formulation for our model (2.10 - 2.13) is equivalent to that of the Cryer problem (4.2 - 4.5). Following that, we comment on modifications needed to implement the above-mentioned results for our model, and the choice of material parameters that best represent the incompressible constituent limits. After that, we wrap up the chapter with the comparison of the results.

4.2.1 Deriving Cryer Problem from Our Model

To reproduce the conditions of the Cryer problem, we assume that there is no gravitational field present; thus, we consider the gravity acceleration g to be zero and, to satisfy the boundary condition for stress, set the value of the tidal potential B as a constant (4.4)

$$B = \frac{P_0}{\rho_0} H(t - 0). \quad (4.12)$$

Therefore, all its spatial derivatives are null

$$\nabla B = \vec{0}, \quad (4.13)$$

$$\Delta B = 0. \quad (4.14)$$

First, we deal with the boundary conditions. Since our model is formulated in increments, there is no excess pressure outside the ball, henceforth the permeable boundary condition (2.13) becomes

$$p = 0 \quad \text{at } r = r_0. \quad (4.15)$$

The strong formulation of the free surface boundary condition (2.12)

$$\left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] \cdot \vec{n}_b + u_r(\rho_o - \rho)g\vec{n}_b = -\rho_o B \vec{n}_b \quad \text{at } \Gamma$$

under assumptions on the form of the tidal potential B (4.12) and on the absence of the gravity field ($g = 0$) becomes

$$\sigma_{rr} = -P_0 H(t - 0) \quad \text{at } r = r_0 \quad (4.16)$$

because the term in square brackets is equal to the total stress $\boldsymbol{\sigma}$ (2.5) and since the outer unit normal of the sphere is the unit radial vector $\vec{e}_r$. Thus, we have managed to obtain the surface boundary conditions of the Cryer problem (4.4) from our model and hence showed their equivalence. The boundary conditions in the centre (4.5) can be freely added to the system.

With the boundary conditions cleared, we show how from our model setting we obtain the Navier equation (4.2) first and then the pressure diffusion equation (4.3).

The curl of $\vec{u}$ in spherical coordinates is

$$\frac{1}{r \sin \theta} \left(\frac{\partial(u_\varphi \sin \theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \varphi} \right) \vec{e}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial(r u_\varphi)}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \vec{e}_\varphi.$$

Thus, due to the spherical symmetry of $\vec{u}$ (4.1) the displacement field is irrotational, hence

$$\nabla \times \vec{u} = 0. \quad (4.17)$$

To demonstrate its equivalence to the Navier equation (4.2), we consider the displacement equation (2.10)

$$\nabla \cdot \left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] + \rho \nabla B = 0,$$

with $\nabla B = 0$ as per (4.13) it takes the form

$$\nabla \cdot \left[\left((K_u - \alpha^2 M)(\nabla \cdot \vec{u}) - \alpha p \right) \mathbb{I} + 2\mu \mathbb{D}_d(\vec{u}) \right] = 0,$$

using the definition of $\mathbb{D}_d(\vec{u})$ (2.6) and propagating the divergence leads to

$$\left(K_u - \alpha^2 M - \frac{2}{3}\mu \right) \nabla(\nabla \cdot \vec{u}) + \mu \nabla \cdot (\nabla \vec{u} + (\nabla \vec{u})^T) = \alpha \nabla p,$$

propagating the divergence once more gives us

$$\mu \Delta \vec{u} + \left(K_u - \alpha^2 M + \frac{\mu}{3} \right) \nabla(\nabla \cdot \vec{u}) = \alpha \nabla p,$$

employing the following relation between the material constants (Cheng [11], B.5)

$$K_u - \alpha^2 M = \frac{2\mu(1 + \nu)}{3(1 - 2\nu)}$$

results in

$$\mu \Delta \vec{u} + \frac{\mu}{1 - 2\nu} \nabla(\nabla \cdot \vec{u}) = \alpha \nabla p, \quad (4.18)$$

where ν is the *drained Poisson ratio*. The Laplace operator can be expanded as

$$\Delta \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}),$$

therefore, since $\vec{u}$ is irrotational (4.17), the previous equation (4.18) becomes

$$\mu \left(1 + \frac{1}{1 - 2\nu}\right) \nabla(\nabla \cdot \vec{u}) = \alpha \nabla p, \quad (4.19)$$

which, if multiplied by $\frac{\eta}{\mu\alpha}$ finally yields

$$\nabla(\nabla \cdot \vec{u}) = \frac{\eta}{\mu} \nabla p, \quad (4.20)$$

thanks to the definition of *poroelastic stress coefficient* η (Cheng [11], B.11)

$$\eta = \frac{\alpha(1 - 2\nu)}{2(1 - \nu)},$$

it holds that

$$\frac{\eta}{\mu\alpha} \mu \left(1 + \frac{1}{1 - 2\nu}\right) = \frac{\eta}{\alpha} \frac{1 - 2\nu + 1}{1 - 2\nu} = \eta \frac{2(1 - \nu)}{\alpha(1 - 2\nu)} = \frac{\eta}{\eta} = 1.$$

This proves the equivalence to the Navier equation (4.2)

$$\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial r^2 u_r}{\partial r} \right) = \frac{\eta}{\mu} \frac{\partial p}{\partial r},$$

as its formulation in invariants is the same as equation (4.20) we derived

$$\nabla(\nabla \cdot \vec{u}) = \frac{\eta}{\mu} \nabla p. \quad (4.21)$$

To show the equivalence to the pressure diffusion equation (4.3), we take the pore pressure equation (2.11)

$$\frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t} - \frac{\kappa}{\eta_f} \Delta p + \frac{\kappa}{\eta_f} \rho_f^m \Delta B = 0$$

with $\Delta B = 0$ (4.13) reducing it to

$$\frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t} - \frac{\kappa}{\eta_f} \Delta p = 0,$$

and change the terms' sides

$$\frac{1}{M} \frac{\partial p}{\partial t} - \frac{\kappa}{\eta_f} \Delta p = -\alpha \frac{\partial(\nabla \cdot \vec{u})}{\partial t}. \quad (4.22)$$

To simplify the equation, our aim is to eliminate the divergence of displacement $\vec{u}$. For that purpose, direct spatial integration of the invariant form of the Navier equation (4.21) yields

$$\nabla \cdot \vec{u} = \frac{\eta}{\mu} p + C(t),$$

where $C(t)$ is an integration constant dependent only on time. Substituting this result into the previous equation (4.22) and grouping the derivatives gives

$$\left(\frac{1}{M} + \frac{\alpha\eta}{\mu} \right) \frac{\partial p}{\partial t} - \frac{\kappa}{\eta_f} \Delta p = -\alpha \frac{\partial C(t)}{\partial t}. \quad (4.23)$$

We define the *storage coefficient* S (Cheng [11], B.14)

$$S = \frac{1}{M} + \frac{\alpha\eta}{\mu}$$

and then divide the equation (4.23) with it to get

$$\frac{\partial p}{\partial t} - \frac{\kappa}{S\eta_f} \Delta p = -\frac{\alpha}{S} \frac{\partial C(t)}{\partial t}.$$

Setting the *consolidation coefficient* c as (Cheng [11], B.16)

$$c = \frac{\kappa}{S\eta_f}$$

results in

$$\frac{\partial p}{\partial t} - c \Delta p = -\frac{\alpha}{S} \frac{\partial C(t)}{\partial t}.$$

Finally, we can express the integration constant $C(t)$ as

$$C(t) = 3A_1(t),$$

to arrive at

$$\frac{\partial p}{\partial t} - c \Delta p = -\frac{3\alpha}{S} \frac{\partial A_1(t)}{\partial t},$$

which is the sought pressure diffusion equation (4.3).

And with that, we have proven that the Cryer problem (4.2 - 4.5) is equivalent to the strong formulation of our model (2.10 - 2.13) under the assumptions of sphere symmetry, no gravity field, and the spatially constant form of the tidal potential (4.12). Henceforth, the Cryer problem is a valid choice as a benchmark for our model.

4.2.2 Choice of Parameters and Model Modification

To validate our model and its implementation, we use the Cryer problem as a benchmark. Specifically, we want to compare the analytical solution of the benchmark (4.6) with the numerical solution obtained from our formulation.

As shown in the preceding subsection, the following changes to the code have to be implemented to ensure the correspondence of our model to the Cryer problem.

- The spatial derivatives of the tidal potential B are null (4.13) – the right-hand side of the monolithic equation (3.8) contains only the penalty term of the Nitsche method (3.6).
- The boundary conditions take the form of (4.4). Therefore:
 - The boundary integral in the monolithic equation (3.8) originating from the free surface boundary condition contains only the constant loading pressure (4.16).
 - The ocean pressure P_0 is set to zero according to (4.15).

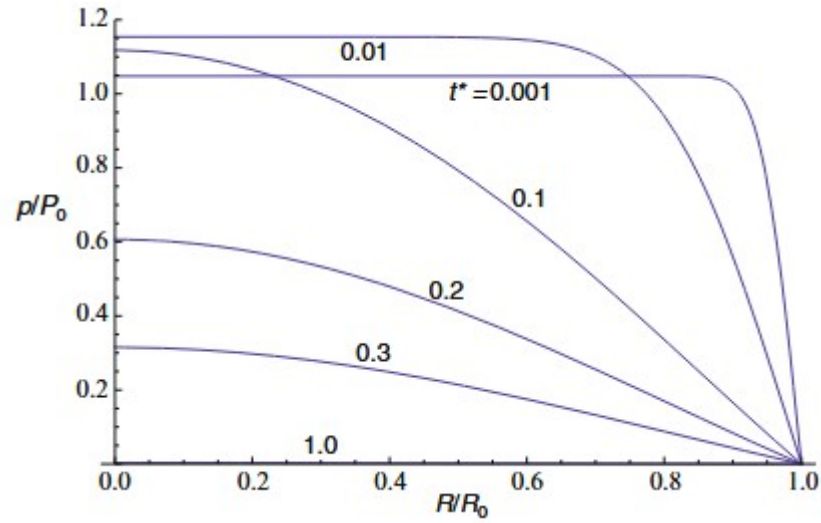


Figure 4.1 “Normalized pressure distribution for the Cryer problem at various times, for an incompressible constituent model with $\nu = 0.25$ ”

The following values of material constants are used in the code and correspond to the case where the porous medium and fluid are taken to be *Hard sediment* and *Water*, respectively.

- Shear modulus – $\mu = 2.61 \times 10^7$ Pa
- Biot effective stress coefficient – $\alpha = 0.999$
- Porosity – $\Phi = 0.47$
- Intrinsic permeability – $\kappa = 10^{-10}$ m²
- Dynamic viscosity of the fluid – $\eta_f = 10^{-3}$ Pa · s
- Bulk modulus of the Hard sediment – $K_s = 3.6 \times 10^{10}$ Pa
- Bulk modulus of the Water – $K_f = 2.25 \times 10^9$ Pa
- Sphere radius – $r_0 = 192$ km

The reference for all but two parameters is Cheng [11], with one exception being the Biot effective stress coefficient α with references Abousleiman et al. [25] and Dalrymple and Liu [26] and the other being the sphere radius r_0 with same value as the core radius of Enceladus Čadek et al. [3].²

The material constants are selected to emulate the incompressible constituent limit as closely as possible, with the drained Poisson ratio ν approaching the value 0.25. This choice is motivated by the fact that, under these conditions, the distribution of the analytical solution for the normalised pressure (4.6) plotted at various dimensionless times t^* (4.10) clearly exhibits the Mandel-Cryer effect (see Figure 4.1 for the limit case presented in Cheng [11]). This is discussed in more detail in the next section, which focuses on model validation through comparison to the numerical solution corresponding to the above-mentioned code modifications and parameter values.

4.2.3 Results Comparison

To see if the model and the code were properly derived and implemented, we compare analytical and numerical solutions corresponding to the same values of dimensionless time t^* . Five different values of dimensionless time $t^* \{0.001, 0.01, 0.1, 0.2, 0.3\}$ are considered. All numerical solutions are computed with appropriate time period in 100 steps, where the solution in the last last step represents the chosen dimensionless time. The time end time t is calculated using the inverse of the definition of t^* (4.10) as

$$t = \frac{r_0^2 t^*}{c}.$$

These times allow us to observe the desired Mandel-Cryer effect of the pressure rising above its initial value at small times. The presence of this phenomenon is an important criterion for assessing the validity of our model. At later times, the pressure should decrease as the fluid begins to redistribute and drain. The numerical solution is computed according to the changes and parameters described in the previous section, and the analytical solution is computed with the use of *NumPy* (Harris et al. [27]) and *SciPy* (Virtanen et al. [28]) Python libraries. The results are visualised with the *Matplotlib* (Hunter [29]) Python library.

For comparison, on one graph we plot both solutions for all values of t^* as the distribution of normalised pressure $\frac{p}{p_0}$ as a function of the proportional radial distance from the centre to the boundary $\frac{r}{r_0}$. The colours distinguish numerical solutions at different dimensionless times. The crosses represent values at degrees of freedom (DoF) – nodes of the mesh over which the solution was computed. The solid coloured lines then serve as function approximations of those values. Validation of the model can be done against the analytical solutions displayed using the dashed black lines.

The first result (Figure 4.2) is for the mesh with a uniform layering (20 layers; total number of cells: 65,948). The results presented in the following chapter are computed using the same mesh. We see that at all layers of the mesh the numerical solutions are very close to the analytical ones for all but one values of t^* . The sole exception being the dimensionless time 0.001 showing greater discrepancies

²Other physical characteristics of Enceladus can be found in the table 5.1

near the boundary. We argue that this is due to the uniformity of the mesh, as then it is not fine enough to properly manage the impact of the loading condition (4.16) in such a short time interval. We prove this claim by using a finer mesh (increased resolution near the boundary; total number of cells: 131,731), as then the numerical solutions no longer exhibit this behaviour (Figure 4.3).

We thus consider our model and code successfully validated against the Cryer problem as a benchmark since we have managed to show that our numerical solutions closely approximate the analytical ones and also manage to properly capture the Mandel-Cryer effect.

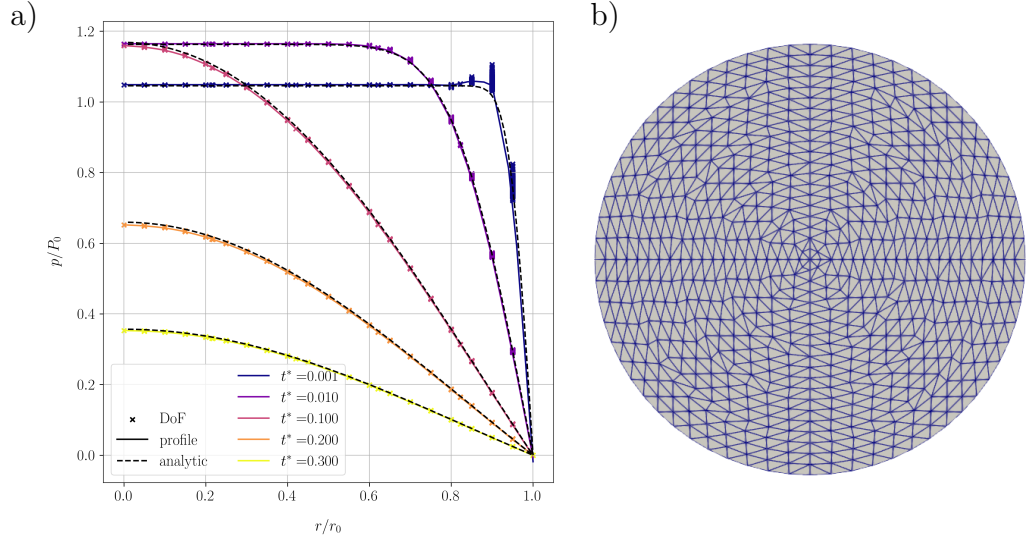


Figure 4.2 Solution of the Cryer problem at times $t^* = 0.001, 0.01, 0.1, 0.2, 0.3$. **(Panel a)** — Colored lines: numerical solution (depth-averaged); colored points: values at degrees of freedom; dashed line: analytical solution. **(Panel b)** — Corresponding computational mesh (equatorial cross-section) with uniform vertical resolution (20 layers); total number of cells: 65,948.

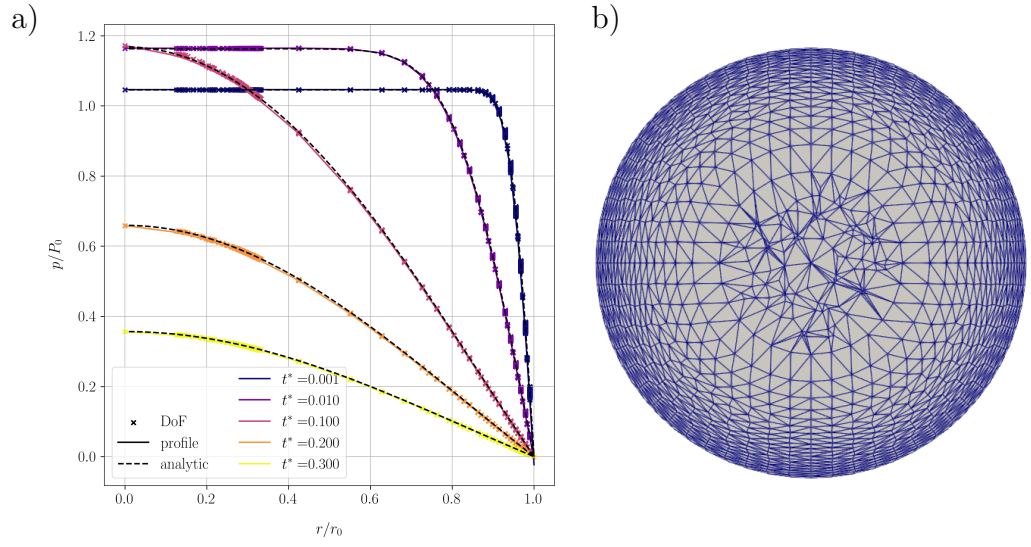


Figure 4.3 Solution of the Cryer problem at times $t^* = 0.001, 0.01, 0.1, 0.2, 0.3$. **(Panel a)** — Colored lines: numerical solution (depth-averaged); colored points: values at degrees of freedom; dashed line: analytical solution. **(Panel b)** — Corresponding computational mesh (equatorial cross-section) with increasing resolution towards the surface; total number of cells: 131,731.

5 Applications to Tidal Deformation

With our model and code validated, we now dedicate this chapter to the computation and discussion of the numerical results and using the full system for the poroelastic model (3.8 - 3.10) applied to Enceladus's porous core. We first introduce the default parameter set for the reference solution and the computation settings which are used for the computation of all the solutions. We then explore the effect of two material constants, intrinsic permeability κ and Biot effective stress coefficient α . For both of them, we compare the reference solution with three other solutions. Two of them are similar to the reference solution, differing only in the value of the studied parameter, one larger and the other smaller than in the reference case. The last solution corresponds to the purely elastic setting computed using the baseline code (Souček et al. [18]) with a shared subset of parameter values from the reference solution. The comparison shows the pore pressure p and the displacement u at various times corresponding to Enceladus's position along its orbit. This allows us to comment on the effect of chosen material parameters, highlighting their influence on the displayed variables and their role in the presence of differences between purely elastic and poroelastic models.

5.1 Default Settings

Here we present the numerical settings of the code and all parameters used for the computation of the reference elastic and poroelastic numerical solutions: form of the tidal potential B , values of material constants and physical characteristics of Enceladus.

Tidal potential B assuming a synchronously rotating body (neglecting terms $\mathcal{O}(e^2)$)

$$B(r, \theta, \varphi, t) = -r^2 n^2 e \times \left\{ \frac{3}{2} P_2^0(\cos \theta) \cos nt - \frac{1}{4} P_2^2(\cos \theta) [3 \cos nt \cos 2\varphi + 4 \sin nt \sin 2\varphi] \right\}$$

where r , θ , and φ are the *radius*, *co-latitude* and *east-positive longitude*; $n = \frac{2\pi}{P}$ denotes the *angular frequency*, P is the *rotational and orbital period*; e is the *eccentricity*; P_2^0 and P_2^2 are the associated *Legendre functions*.

Default values of model parameters:

- Shear modulus – $\mu = 10 \times 10^9$ Pa
- Biot effective stress coefficient – $\alpha = 0.9$
- Porosity – $\Phi = 0.2$
- Intrinsic permeability – $\kappa = 10^{-7}$ m²

- Dynamic viscosity of the fluid – $\eta_f = 1.9 \times 10^{-3} \text{ Pa} \cdot \text{s}$
- Bulk modulus of the solid – $K_s = 16.8571 \times 10^9 \text{ Pa}$
- Bulk modulus of the fluid – $K_f = 2.2 \times 10^9 \text{ Pa}$
- Additional Enceladus’ physical properties are listed in Table 5.1

All values but one are consistent with the reference Rovira-Navarro et al. [9], the only exception being the solid bulk modulus K_s , calculated from the bulk modulus of the fluid and *Poisson’s ratio* $\left(\frac{107}{424}\right)$. The reference solution for the purely elastic case does not use parameters which are not part of that model.¹

core radius r_0	192 kg	Čadek et al. [3]
core density ρ	2366 kg m ⁻³	Čadek et al. [3]
ocean density ρ_o	1010 kg m ⁻³	Čadek et al. [3]
eccentricity e	0.0047	Jacobson [30]
orbital period P	1.370 218 d	Jacobson [30]

Table 5.1 Physical characteristics of Enceladus.

Numerical settings: To obtain results, we compute five periods each consisting of 100 time steps and take the solution from the last period because by then, the periodicity of it is established. For the results below, we tested computations utilising more time steps per period (1000) or more periods (50), but in neither case have we found any discrepancies between solutions. Therefore, we claim that our choice of numerical parameters (5 periods, 100 steps each) is satisfactory and has no negative impact on either the results presented or the conclusions made. We used a computational mesh with uniform vertical resolution (20 layers) and a total number of cells 65,948 (5.1). It is the same as the one for the Cryer problem (4.2).

¹In particular Biot effective stress coefficient α , porosity Φ , intrinsic permeability κ and dynamic viscosity of the fluid η_f .

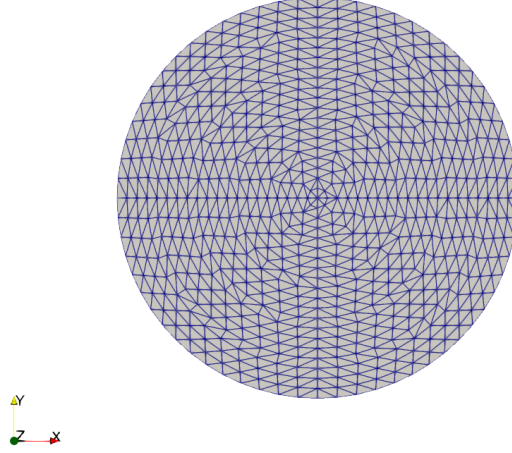


Figure 5.1 Uniform mesh (layers: 20; total number of cells: 65948)

5.2 Results Commentary

To examine the corresponding poroelastic effects, we compute two additional sets of solutions: one focusing on the intrinsic permeability κ , and the other on the effective stress coefficient α . Each new solution differs from the above-mentioned default settings only in the value of the parameter studied. In particular, for κ we use the values of 10^{-6} and 10^{-8} in place of the reference value 10^{-7} , and for α we consider values of 0.8 and 0.99 instead of the default value 0.9. For each parameter analysed, we display these two solutions together with the reference ones for the poroelastic and purely elastic models, and comment on differences present. We visualise the displacement $\vec{u}$ and the pore pressure p , at three orbital phases: pericentre² ($t = 0P$, first row), quarter-period ($t = 0.25P$, second row), and apocentre³ ($t = 0.5P$, third row).

The dependent material constants for different parameter values are listed in Table 5.2. The poroelastic diffusion time t_d normalised by the orbital period P is close to 1 except for the alternate values of κ , but as mentioned before, we have verified that the numerical parameters are set correctly and do not spoil these results.⁴

The poroelastic diffusion time t_d is calculated from its definition (4.11) together with the rest of the dependent constants using the following relations from Cheng [11]:

²point of closest approach

³point of farthest approach

⁴Tested: $\kappa = 1 \times 10^{-8} \text{ m}^2$, $\alpha = 0.9$: 100 time steps per period, 50 periods (5x poroelastic diffusion time)

Tested: $\kappa = 1 \times 10^{-6} \text{ m}^2$, $\alpha = 0.9$: 1000 time steps per period, 2 periods (at least 100 steps per poroelastic diffusion time)

$$K = (1 - \alpha) K_s \quad (\text{B. 39})$$

$$K_u = 1 \quad (\text{B. 38})$$

$$M = \frac{K_u - K}{\alpha^2} \quad (\text{B. 13})$$

$$\eta = \frac{\alpha(1 - 2\nu)}{2(1 - \nu)} \quad (\text{B. 11})$$

$$S = \frac{1}{M} + \frac{\alpha\eta}{\mu} \quad (\text{B. 14})$$

$$c = \frac{\kappa}{S\eta_f} \quad (\text{B. 16})$$

α	κ m ²	S Pa ⁻¹	c m ² s ⁻¹	t_d s	t_d/P	M GPa	K_u GPa	K GPa
0.90	10 ⁻⁶	1.70 × 10 ⁻¹⁰	3.1 × 10 ⁶	1.2 × 10 ⁴	0.102	7.6	7.8	1.7
0.90	10 ⁻⁷	1.70 × 10 ⁻¹⁰	3.1 × 10 ⁵	1.2 × 10 ⁵	1.02	7.6	7.8	1.7
0.90	10 ⁻⁸	1.70 × 10 ⁻¹⁰	3.1 × 10 ⁴	1.2 × 10 ⁶	10.2	7.6	7.8	1.7
0.99	10 ⁻⁷	1.90 × 10 ⁻¹⁰	2.8 × 10 ⁵	1.3 × 10 ⁴	1.14	7.4	7.5	0.84
0.80	10 ⁻⁷	1.60 × 10 ⁻¹⁰	3.3 × 10 ⁶	1.1 × 10 ⁴	0.957	7.9	8.4	3.4

Table 5.2 Material constants computed with default settings for selected values of Biot effective stress coefficient α and intrinsic permeability κ .

5.2.1 Intrinsic Permeability κ

The solutions for this subsection are shown in Figure 5.2, one column for each solution. As we see in Table 5.2, the choice of κ affects the poroelastic diffusion time t_d , which is an important factor for the presence of poroelastic effects. As the tidal potential B depends quadratically on the radial distance r , the tidal force is largest at the surface and equal to zero in the centre. As a result of this, the solutions corresponding to decreasing values of intrinsic permeability κ exhibit the skin effect – the complex behaviour concentrates to the uppermost parts of the core.

When the diffusion time is shorter than the tidal period (fourth column; $\kappa = 10^{-6}$), we see no significant effect on the pore pressure because the fluid motion is fast and the fluid has enough time to react to changes in the tidal force. If the diffusion time is comparable (third column; $\kappa = 10^{-7}$) or larger (second column; $\kappa = 10^{-8}$) than the tidal period, the fluid does not have enough time to react and as the pore pressure pattern becomes more complex, we can observe the pore pressure starting to lag behind the applied forcing and the elastic response.

With increasing value of κ , the deformation of the core also increases as the contribution of the fluid to the bulk modulus of the core diminishes as the fluid is able to redistribute quickly. The bulk modulus of the core for a smaller value

of intrinsic permeability (fourth column; $\kappa = 10^{-6}$) is closer to $K = 1.7$ GPa (drained limit), while for a larger value (second column; $\kappa = 10^{-8}$) it approaches $K_u = 7.8$ GPa (undrained limit), which is closer to the bulk modulus of the solid ($K_s = 16.8571$ GPa). The change in deformation can be observed in the comparison Figure 5.2, with the smallest deformation in the elastic case (first column) and the largest deformation in the poroelastic model with the largest permeability (fourth column; $\kappa = 10^{-6}$).

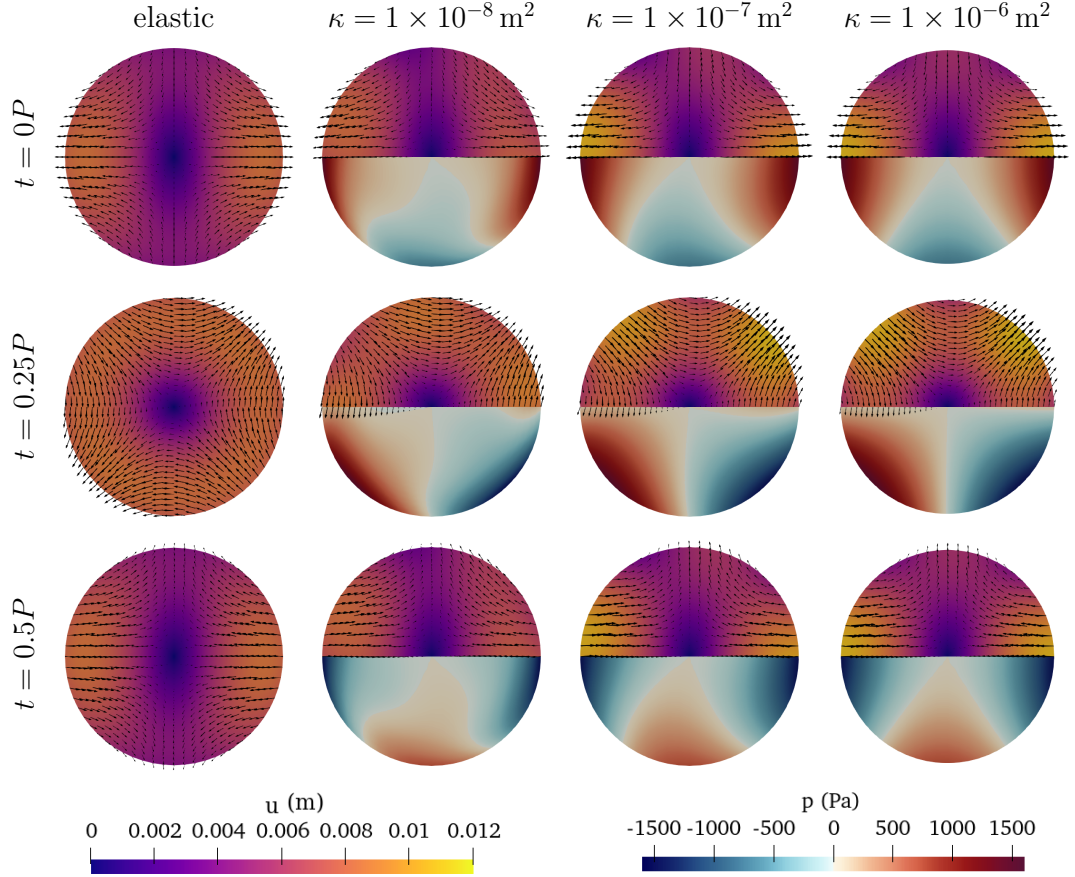


Figure 5.2 Comparison of elastic (first column) and poroelastic models (columns two to four) at three orbital phases: pericentre ($t = 0P$, first row), quarter-period ($t = 0.25P$, second row), and apocentre ($t = 0.5P$, third row). The top half of each circle shows displacement: color indicates magnitude, and arrows represent direction. The bottom half shows pore pressure p . Model parameters: Biot effective stress coefficient $\alpha = 0.9$; intrinsic permeability $\kappa = 1 \times 10^{-8} \text{ m}^2$, $1 \times 10^{-7} \text{ m}^2$, and $1 \times 10^{-6} \text{ m}^2$. The depicted equatorial cross-section has the x -axis oriented to the right and the y -axis oriented upward.

5.2.2 Biot Effective Stress Coefficient α

The solutions for this subsection are shown in Figure 5.3. Increasing the value of Biot Effective Stress Coefficient α promotes poroelastic behaviour, as it lowers the value of bulk moduli of the core (K_u and K , see Table 5.2), thus increasing the magnitude of deformation. Same as in the case of permeability, the smallest deformation in is the elastic case (first column) and the largest deformation in

the poroelastic model with the largest Biot Effective Stress Coefficient α (fourth column; $\alpha = 0.99$) with the increase in the two other poroelastic solutions (second and third column; $\alpha = 0.8$ and $\alpha = 0.9$, respectively).

The most interesting result is for the value $\alpha = 0.99$ (fourth column) with the diffusion time corresponding to the reference value of permeability ($\kappa = 10^{-7}$) is comparable to the tidal period P , since we observe changes in the deformation pattern. We notice the deformation seemingly ahead of both the elastic response and the applied forcing.

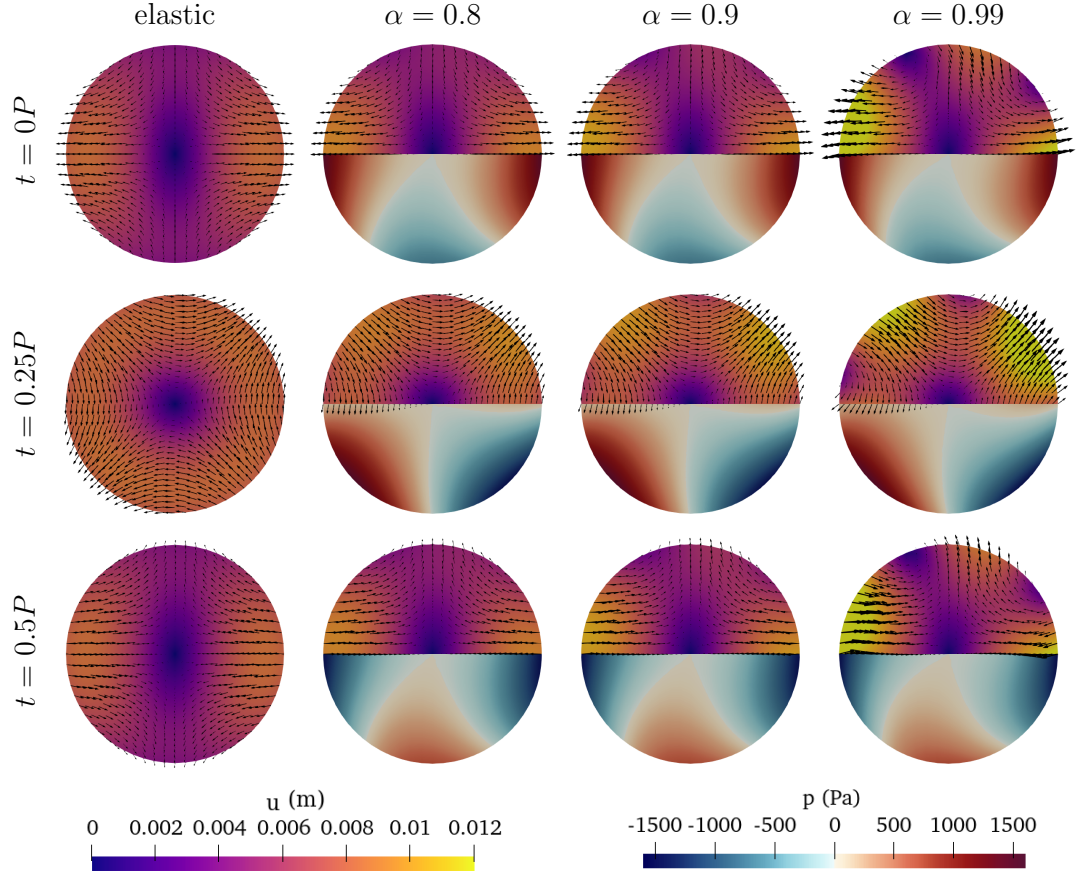


Figure 5.3 Comparison of elastic (first column) and poroelastic models (columns two to four) at three orbital phases: pericentre ($t = 0P$, first row), quarter-period ($t = 0.25P$, second row), and apocentre ($t = 0.5P$, third row). The top half of each circle shows displacement: color indicates magnitude, and arrows represent direction. The bottom half shows pore pressure p . Model parameters: intrinsic permeability $\kappa = 1 \times 10^{-7} \text{ m}^2$; Biot effective stress coefficient $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 0.99$. The depicted equatorial cross-section has the x -axis oriented to the right and the y -axis oriented upward.

Conclusion

In this thesis, we have derived the model and developed numerical software for the poroelastic deformation with applications to Enceladus's core.

The poroelastic model was introduced as a two-component mixture of a porous solid matrix with fluid-filled pores. The introduction also contained information on the specific concepts of the model and the new variables. We described it with a set of five governing equations with four independent material constants. Two equations were based on constitutive theory and three others on physical laws, namely the balance of mass, the balance of momentum, and the Darcy law.

To obtain the system for the core model, the set of governing equations was accompanied by two boundary conditions. One described the free surface condition and the other the permeable property of the core. We presented strong and weak formulations for the system and finally formulated it using a single monolithic partial differential equation with pore pressure and displacement as variables. This kind of formulation allowed us to numerically solve the problem as one equation of two variables instead of having to solve two equations of two coupled variables each.

The monolithic system was later discretised in time and space to allow numerical solution based on the Galerkin approximation and finite element method. As part of this process, we also used the Nitsche method to weakly impose the Dirichlet boundary condition.

This was then implemented in the code, extending the capability of the base solution for the elastic case. This implementation and the derived system were validated against the Cryer problem chosen as the benchmark. We showed the equivalence of problem formulations under certain justified assumptions. We commented on necessary changes to reflect the benchmark setting and compared our numerical solution against the analytical one of the Cryer problem. The validation was successful, with numerical solutions closely following the analytical one, and we also managed to properly replicate the presence of the Mandel-Cryer effect of the benchmark problem.

With our model and code properly tested, we discussed the effect of certain poroelastic material constants on the periodic behaviour of the core under the influence of tidal forcing in comparison to the purely elastic model. We have managed to show how value of permeability and Biot effective stress coefficient affect the distribution of the pore pressure and the displacement. We noted that our model captures higher magnitudes of deformation than the purely elastic one, which corresponds to the physics. We also highlighted the effect of values of Biot effective stress close to one on the deformation that precedes the deformations from the reference solutions.

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