

Weakly nonlinear analysis of Rayleigh–Bénard convection problem in extended Boussinesq approximation



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ARTICLE INFO

Article history:

Received 24 September 2020

Accepted 12 May 2021

2010 MSC:

76E06

76E20

80A20,

Keywords:

Thermal convection

Extended boussinesq approximation

Weakly nonlinear analysis

ABSTRACT

We investigate Rayleigh–Bénard convection problem in an extended Boussinesq approximation suitable for conditions in the Earth's mantle. The aim is to evaluate the influence of depth-dependent material parameters, dissipation, adiabatic heating/cooling and heat sources on qualitative characteristics of thermal convection. We identify the critical values of dimensionless parameters that determine the onset of convection, and we characterize the dominating convection patterns in marginally supercritical states. These issues are addressed by the application of linear stability analysis and weakly nonlinear analysis. We have found that the character of convection differs substantially from the standard case of Rayleigh–Bénard convection.

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1. Introduction

The classical system of governing equations for thermal convection in a horizontal layer heated from below reads

$$\operatorname{div} \mathbf{v} = 0, \quad (1.1a)$$

$$\frac{1}{\operatorname{Pr}} \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\operatorname{Ra}(\theta - \theta_0) \mathbf{e}_z - \nabla \pi + \Delta \mathbf{v}, \quad (1.1b)$$

$$\frac{\partial \theta}{\partial t} = \Delta \theta - \mathbf{v} \cdot \nabla \theta, \quad (1.1c)$$

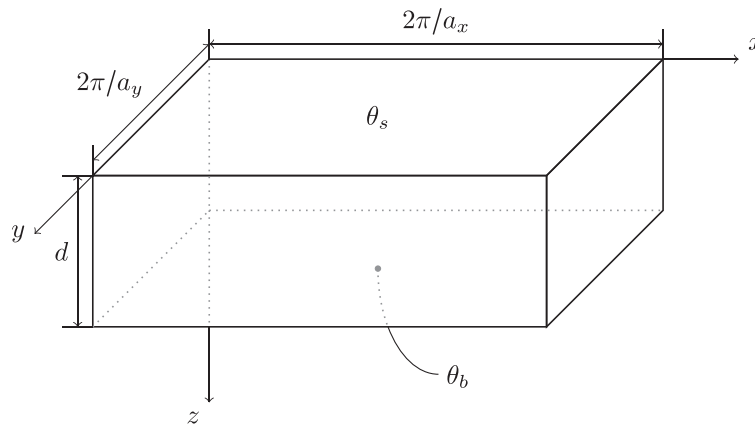
see for example [1] or [2]. In (1.1) $\mathbf{v}$ denotes the velocity, θ denotes the temperature, and $\pi = p - p_0$ where p denotes pressure and p_0 denotes the reference hydrostatic pressure. The reference temperature is denoted by θ_0 , and Pr and Ra denote the Prandtl and Rayleigh number, respectively. This system is derived using the *classical Boussinesq approximation*, which assumes that the material coefficients – thermal conductivity, thermal expansivity and dynamic viscosity – are constant. Moreover, the classical derivation neglects dissipation, adiabatic heating/cooling and heat sources.

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¹ Mark Dostálík has been supported by the Charles University SVV project No. 260577.

² Vít Průša thanks the Czech Science Foundation project 20-11027X for its support.

Fig. 1. Periodic cell Ω .

In various settings, most notably in geophysical applications, material properties such as thermal conductivity, thermal expansivity and dynamic viscosity are pressure- and temperature-dependent, which makes the classical Boussinesq approximation inapplicable, and the governing equations must be changed accordingly. (See for example [3,4] for a discussion regarding pressure/temperature dependent material parameters.) The same holds if the fluid of interest cannot be described by the classical Navier–Stokes–Fourier model with variable coefficients and a more complex non-Newtonian fluid model must be employed. (See for example [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] and [15], for analysis of Rayleigh–Bénard convection problem for viscoelastic fluids, shear-thinning fluids, n -grade fluids and viscoplastic fluids.) In the present contribution we investigate the Rayleigh–Bénard convection problem using the *extended Boussinesq approximation*, which assumes *depth-dependent material coefficients* and also retains the effects of *dissipation, adiabatic heating/cooling and heat sources*, which is a setting relevant, for example, in geophysics.

More specifically, we aim at the identification of the critical values of dimensionless parameters that determine the onset of convection, and the characterization of dominating convection patterns in marginally supercritical states. These issues are addressed by the application of linear stability analysis, where we follow the classical linearized stability theory, see, for example, [1], while the characterization of the dominating flow patterns is investigated using the methods of weakly nonlinear analysis, see, for example, [16] and [17,18].

2. Problem description

Regarding the problem geometry, the setting we are interested in is the classical setting for thermal convection, see Section 2.1. The main departure from the classical Rayleigh–Bénard convection problem consists in the use of *extended Boussinesq approximation* that brings additional terms to the right-hand side of the temperature evolution equation. Namely, we take into account the dissipative heating and internal heat sources, and we also consider depth-dependent material parameters. The extended Boussinesq approximation is introduced in Section 2.2 and Section 2.3. Note that the presence of internal heat sources and/or depth-dependent thermal conductivity changes the reference stationary temperature profile, see Section 2.5, hence unlike in the classical case we, in some settings, need to work with a nonlinear reference stationary temperature profile. The boundary conditions are chosen to be the classical ones, see Section 2.4, and the evolution equations for velocity/temperature perturbations, that are used in the remaining part of the text, are formulated in Section 2.6.

2.1. Geometry

We are, in principle, dealing with an unbounded horizontal layer of a fluid but it is customary to consider a bounded subdomain, say Ω , often called the *periodic cell*. We assume that this cell is a parallelepiped as depicted in Fig. 1. Here, d is the height of the fluid layer, a_x , a_y are the wavenumbers corresponding to the x , y coordinates, respectively, and θ_s , θ_b represent the temperature of the top and the bottom boundaries, respectively. On the side walls we prescribe periodic boundary conditions for all the quantities while on the top and the bottom boundaries we prescribe boundary conditions specified in Section 2.4. Note that the original unscaled quantities are used in Fig. 1 as opposed to the following.

2.2. Governing equations

We work with the so-called *extended Boussinesq approximation*, see [2] for detailed derivation and the scaling. (For a discussion of thermal convection problem with dissipative heating see also [19], and for general remarks and references regarding the Boussinesq approximation we refer the reader to [20] and [21].) The governing dimensionless equations read

$$\operatorname{div} \mathbf{v} = 0, \quad (2.1a)$$

$$\frac{1}{\operatorname{Pr}} \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\operatorname{Ra} \alpha (\theta - \theta_0) \mathbf{e}_z - \nabla \pi + \operatorname{div} (2\mu \mathbb{D}), \quad (2.1b)$$

$$\frac{\partial \theta}{\partial t} = \operatorname{div} (k \nabla \theta) - \mathbf{v} \cdot \nabla \theta + \operatorname{Di} \alpha \left(\theta + \frac{\Xi}{\operatorname{Ra}} \right) \mathbf{v} \bullet \mathbf{e}_z + \frac{\operatorname{Di}}{\operatorname{Ra}} 2\mu \mathbb{D} : \mathbb{D} + \frac{\operatorname{Ra}_q}{\operatorname{Ra}}, \quad (2.1c)$$

where $\mathbf{v}$ is the velocity field, $\mathbb{D} =_{\text{def}} \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^\top)$ denotes the symmetric part of the velocity gradient, θ denotes temperature, θ_0 denotes the reference hydrostatic temperature, $\pi = p - p_0$ where p is pressure and p_0 is the reference hydrostatic pressure. Symbols k , α , μ denote dimensionless thermal conductivity, thermal expansivity and dynamic viscosity, respectively, scaled by their surface values, see also Section 2.3. The dimensionless numbers—the *Prandtl number* Pr , the *Rayleigh number* Ra , the *Rayleigh number for heat sources* Ra_q , the *dissipation number* Di , and an auxiliary dimensionless parameter Ξ —are introduced through the relations

$$\operatorname{Pr} = \frac{\mu_s}{\varrho_0 \kappa_s}, \quad (2.2a)$$

$$\operatorname{Ra} = \frac{\varrho_0 \alpha_s (\theta_b - \theta_s) g_0 d^3}{\mu_s \kappa_s}, \quad (2.2b)$$

$$\operatorname{Ra}_q = \frac{\varrho_0 \alpha_s g_0 Q d^5}{\mu_s \kappa_s k_s}, \quad (2.2c)$$

$$\operatorname{Di} = \frac{\alpha_s g_0 d}{c_p}, \quad (2.2d)$$

$$\Xi = \frac{\varrho_0 \alpha_s \theta_s g_0 d^3}{\mu_s \kappa_s}, \quad (2.2e)$$

where d denotes the depth of the convection layer, ρ_0 denotes the (constant) reference hydrostatic density, g_0 denotes the (constant) gravity acceleration, c_p denotes the (constant) specific isobaric heat, Q denotes the (constant) heat sources, and $\kappa = \frac{k}{\rho_0 c_p}$ denotes the thermal diffusivity. The subscript s/b denotes the surface/bottom values of the corresponding quantities.

2.3. Material properties

For the sake of simplicity, we consider the depth-dependent material coefficients k , α , μ in the following simple form

$$k(z) = 1 + b_k z, \quad (2.3a)$$

$$\alpha(z) = \exp(-b_\alpha z), \quad (2.3b)$$

$$\mu(z) = \exp(b_\mu z), \quad (2.3c)$$

where z is the dimensionless depth scaled by d . Moreover, we assume that $b_k \in [1, 10]$, $b_\alpha \in [0.5, 2]$, and $b_\mu \in [1, 4.5]$. This behaviour of material coefficients should (very roughly) approximate the Earth's mantle's material properties, see [22], and gives us a range of parameters that can be met in a real world application. (Note that the depth-dependent material coefficient can be understood as rough approximations of pressure-dependent material coefficients for which we take into account only the dominant contribution of the hydrostatic pressure. For example, the exponential dependence of viscosity on the pressure is known for a long time, see [23], and such models are frequently used in engineering practice, see for example [24]. For further discussion we refer the reader to [3,4].) In order to investigate the influence of the individual coefficients on the onset and qualitative character of the convection, we always consider all the material coefficients k , α , μ being constant except for one being a depth-dependent function.

In the following we assume that all the constant material coefficients appearing in (2.2) are fixed, except for the bottom temperature θ_b . Varying θ_b thus modifies Rayleigh number Ra solely and all the remaining dimensionless numbers Pr , Ra_q , Di , and Ξ stay fixed. We mostly set $\operatorname{Pr} = 10^9$, $\operatorname{Di} = 0.5$, $\operatorname{Ra}_q = 10^6$, and $\Xi = 8 \times 10^5$, these values are again a rough approximate of the real conditions in the Earth's mantle, see [2] or [22]. We also consider the cases when $\operatorname{Ra}_q = 0$ or $\operatorname{Di} = 0$.

2.4. Boundary conditions

On the top and the bottom boundaries of the unit cell Ω we prescribe the *no-penetration* boundary condition for the velocity field $\mathbf{v}$

$$\mathbf{v} \cdot \mathbf{n}|_{z=0,1} = 0, \quad (2.4a)$$

together with the *free-slip* boundary condition (a prevalent boundary condition in geophysics)

$$\mathbb{T}\mathbf{n} - (\mathbb{T}\mathbf{n} \cdot \mathbf{n})\mathbf{n}|_{z=0,1} = 0, \quad (2.4b)$$

where $\mathbb{T}$ denotes the Cauchy stress tensor and $\mathbf{n}$ is the outer unit normal vector to the boundary.

In our case $\mathbf{n} = \pm \mathbf{e}_z$ and $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$ which, with the use of the incompressibility condition (2.1a), yields

$$v^z|_{z=0,1} = 0, \quad \left. \frac{\partial^2 v^z}{\partial z^2} \right|_{z=0,1} = 0. \quad (2.5)$$

The boundary conditions (2.5) then easily imply boundary condition for the vorticity $\eta =_{\text{def}} \text{rot} \mathbf{v}$

$$\left. \frac{\partial \eta^z}{\partial z} \right|_{z=0,1} = 0. \quad (2.6)$$

Concerning the temperature field, we simply prescribe the top and the bottom temperature values

$$\theta|_{z=0} = 0, \quad \theta|_{z=1} = 1. \quad (2.7)$$

2.5. Reference state

The hydrostatic reference state is characterized by the reference velocity $\mathbf{v}_0 \equiv \mathbf{0}$, the reference pressure p_0 , and the reference temperature θ_0 which is assumed to be the function of depth only, i.e. $\theta_0 = \theta_0(z)$. Plugging these into the governing equations (2.1), the heat equation (2.1c) reduces to

$$0 = \text{div}(k \nabla \theta_0) + \frac{\text{Ra}_q}{\text{Ra}}, \quad (2.8)$$

which in turn yields a second order linear differential equation for the reference temperature

$$k(z) \frac{d^2 \theta_0}{dz^2} + \frac{dk}{dz} \frac{d\theta_0}{dz} + \frac{\text{Ra}_q}{\text{Ra}} = 0, \quad (2.9a)$$

with the boundary conditions

$$\theta_0|_{z=0} = 0, \quad \theta_0|_{z=1} = 1. \quad (2.9b)$$

In the case of constant thermal conductivity, $k = 1$, the explicit solution to (2.9) reads

$$\theta_0(z) = -\frac{\text{Ra}_q}{2\text{Ra}} z^2 + \left(1 + \frac{\text{Ra}_q}{2\text{Ra}}\right) z, \quad (2.10)$$

while for the depth-dependent thermal conductivity (2.3a) we obtain

$$\theta_0(z) = \left(1 + \frac{\text{Ra}_q}{\text{Ra} b_k}\right) \frac{\ln(1 + b_k z)}{\ln(1 + b_k)} - \frac{\text{Ra}_q}{\text{Ra} b_k} z. \quad (2.11)$$

Fig. 2 shows a comparison of temperature profiles for various possible settings of thermal conductivity k and the dimensionless ratio $\frac{\text{Ra}_q}{\text{Ra}}$. One can clearly see that a combination of internal heating and increasing thermal conductivity can even result in overheating at the bottom of the layer.

2.6. Perturbation equations

Let us perturb the reference hydrostatic state $[\mathbf{v}_0, p_0, \theta_0]$ by a disturbance denoted by $[\mathbf{v}, p, \vartheta]$. Putting the perturbed solution $[\mathbf{v}, p, \theta_0 + \vartheta]$ into the governing equations (2.1) yields

$$\text{div} \mathbf{v} = 0, \quad (2.12a)$$

$$\frac{1}{\text{Pr}} \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\text{Ra} \alpha \vartheta \mathbf{e}_z - \nabla p + \text{div}(2\mu \mathbb{D}), \quad (2.12b)$$

$$\frac{\partial \vartheta}{\partial t} = \text{div}(k \nabla \vartheta) - \mathbf{v} \cdot \nabla (\theta_0 + \vartheta) + \text{Di} \alpha \left(\theta_0 + \vartheta + \frac{\Xi}{\text{Ra}} \right) \mathbf{v} \cdot \mathbf{e}_z + \frac{\text{Di}}{\text{Ra}} 2\mu \mathbb{D} : \mathbb{D}, \quad (2.12c)$$

where we have used the equation for the reference temperature (2.8) in (2.12c).

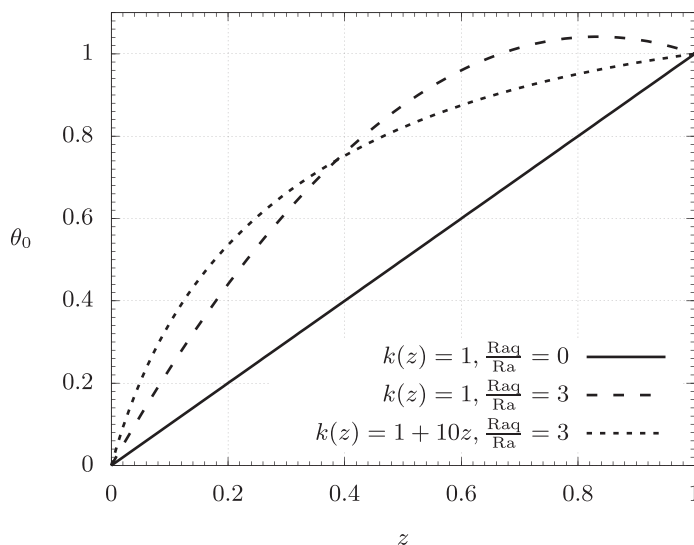


Fig. 2. Reference temperature profiles.

3. Onset of convection

The onset of convection is investigated using the classical linear stability theory, see for example [1].

3.1. Eigenvalue problem formulation

Using the fact that the identity $\Delta \mathbf{v} = -\text{rot rot } \mathbf{v}$ holds for the solenoidal velocity field, see (2.1a), and introducing the normal modes of the velocity disturbance $\mathbf{v}$, and the temperature disturbance ϑ

$$\mathbf{v}(t, \mathbf{x}) = \tilde{\mathbf{v}}(z) \exp [i(a_x x + a_y y) + \sigma t], \quad (3.1a)$$

$$\vartheta(t, \mathbf{x}) = \tilde{\vartheta}(z) \exp [i(a_x x + a_y y) + \sigma t], \quad (3.1b)$$

where $a_x, a_y \in \mathbb{R}$, $\sigma \in \mathbb{C}$, the system of equations (2.12) is subsequently linearized with respect to the perturbation, and it is transformed as follows

$$\frac{\sigma}{\text{Pr}} \left(\frac{d^2}{dz^2} - a^2 \right) \tilde{\vartheta}^2 = a^2 \text{Ra} \alpha \tilde{\vartheta} + \mu \left(\frac{d^2}{dz^2} - a^2 \right)^2 \tilde{\vartheta}^2 + 2 \frac{d\mu}{dz} \left(\frac{d^2}{dz^2} - a^2 \right) \frac{d\tilde{\vartheta}^2}{dz} + \frac{d^2\mu}{dz^2} \left(\frac{d^2}{dz^2} - a^2 \right) \tilde{\vartheta}^2, \quad (3.2a)$$

$$\frac{\sigma}{\text{Pr}} \tilde{\eta}^2 = \mu \left(\frac{d^2}{dz^2} - a^2 \right) \tilde{\eta}^2 + \frac{d\mu}{dz} \frac{d\tilde{\eta}^2}{dz}, \quad (3.2b)$$

$$\sigma \vartheta = k \left(\frac{d^2}{dz^2} - a^2 \right) \tilde{\vartheta} + \frac{dk}{dz} \frac{d\tilde{\vartheta}}{dz} - \frac{d\theta_0}{dz} \tilde{\vartheta}^2 + \text{Di} \alpha \left(\theta_0 + \frac{\Xi}{\text{Ra}} \right) \tilde{\vartheta}^2, \quad (3.2c)$$

where $a^2 =_{\text{def}} a_x^2 + a_y^2$ denotes the wavenumber, and $\tilde{\eta}$ is the z -dependent part of the formula for vorticity

$$\eta(t, \mathbf{x}) = \tilde{\eta}(z) \exp [i(a_x x + a_y y) + \sigma t], \quad (3.3)$$

which can be explicitly derived from (3.1a). System (3.2) represents an eigenvalue problem supplemented with the boundary conditions following easily from (2.5), (2.6), and (2.7)

$$\tilde{\vartheta}^2 \Big|_{z=0,1} = 0, \quad \frac{d^2 \tilde{\vartheta}^2}{dz^2} \Big|_{z=0,1} = 0, \quad \frac{d\tilde{\eta}^2}{dz} \Big|_{z=0,1} = 0, \quad \tilde{\vartheta} \Big|_{z=0,1} = 0. \quad (3.4)$$

As in the classical thermal convection problem, it can be shown by multiplying (3.2b) with the complex conjugate of $\tilde{\eta}^2$, integrating over Ω , and using the Stokes theorem that the disturbance $\tilde{\eta}^2$ is nonzero only if $\text{Re}(\sigma) < 0$. We can thus eliminate (3.2b) from our search for the first eigenvalue with positive real part, and consider the reduced eigenvalue problem

$$\sigma \mathcal{A} \begin{bmatrix} \tilde{\vartheta}^2 \\ \tilde{\vartheta} \end{bmatrix} = \mathcal{B} \begin{bmatrix} \tilde{\vartheta}^2 \\ \tilde{\vartheta} \end{bmatrix}, \quad (3.5a)$$

$$\tilde{v}^z|_{z=0,1} = 0, \quad \left. \frac{d^2 \tilde{v}^z}{dz^2} \right|_{z=0,1} = 0, \quad \tilde{\vartheta}|_{z=0,1} = 0, \quad (3.5b)$$

where we have denoted

$$\mathcal{A} =_{\text{def}} \begin{bmatrix} \frac{1}{\text{Pr}} \left(\frac{d^2}{dz^2} - a^2 \right) & 0 \\ 0 & 1 \end{bmatrix}, \quad (3.6a)$$

$$\mathcal{B} =_{\text{def}} \begin{bmatrix} \mu \left(\frac{d^2}{dz^2} - a^2 \right)^2 + 2 \frac{d\mu}{dz} \left(\frac{d^2}{dz^2} - a^2 \right) \frac{d}{dz} + \frac{d^2 \mu}{dz^2} \left(\frac{d^2}{dz^2} - a^2 \right) & a^2 \text{Ra} \alpha \\ -\frac{d\theta_0}{dz} + \text{Di} \alpha \left(\theta_0 + \frac{\Xi}{\text{Ra}} \right) & k \left(\frac{d^2}{dz^2} - a^2 \right) + \frac{dk}{dz} \frac{d}{dz} \end{bmatrix}. \quad (3.6b)$$

3.2. Numerical solution of the eigenvalue problem

We use the *Chebyshev collocation method* to solve the eigenvalue problem (3.5) numerically. (For more details on the method we refer the reader to [25] or [26].) To simplify incorporating the boundary conditions for $\tilde{v}^z$ we introduce a new variable

$$w =_{\text{def}} \left(\frac{d^2}{dz^2} - a^2 \right) \tilde{v}^z. \quad (3.7)$$

The modified eigenvalue problem thus reads

$$\sigma \mathcal{A}_{\text{mod}} \begin{bmatrix} \tilde{v}^z \\ w \\ \tilde{\vartheta} \end{bmatrix} = \mathcal{B}_{\text{mod}} \begin{bmatrix} \tilde{v}^z \\ w \\ \tilde{\vartheta} \end{bmatrix}, \quad (3.8a)$$

$$\tilde{v}^z|_{z=0,1} = 0, \quad w|_{z=0,1} = 0, \quad \tilde{\vartheta}|_{z=0,1} = 0, \quad (3.8b)$$

where

$$\mathcal{A}_{\text{mod}} =_{\text{def}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\text{Pr}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.9a)$$

$$\mathcal{B}_{\text{mod}} =_{\text{def}} \begin{bmatrix} \left(\frac{d^2}{dz^2} - a^2 \right) & -1 & 0 \\ 0 & \mu \left(\frac{d^2}{dz^2} - a^2 \right) + 2 \frac{d\mu}{dz} \frac{d}{dz} + \frac{d^2 \mu}{dz^2} & a^2 \text{Ra} \alpha \\ -\frac{d\theta_0}{dz} + \text{Di} \alpha \left(\theta_0 + \frac{\Xi}{\text{Ra}} \right) & 0 & k \left(\frac{d^2}{dz^2} - a^2 \right) + \frac{dk}{dz} \frac{d}{dz} \end{bmatrix}. \quad (3.9b)$$

We discretize the differential operators (3.9) using the Matlab function `chebdf`, see [27], which returns the differentiation matrix in Chebyshev points. The eigenvalue problem (3.8) is then solved using the Matlab's implemented function `eig`. The number N of Chebyshev points is set to $N = 40$ for all the performed computations.

In the following subsections we solve the eigenvalue problem (3.8) for constant material parameters except for a single depth-dependent one to evaluate its influence on the shift of the critical wavenumber a_{crit} and Rayleigh number Ra_{crit} compared to the classical thermal convection case. Recall that in the classical case ($k, \alpha, \mu = 1$, $\text{Di} = 0$, $\text{Ra}_q = 0$) we obtain

$$a_{\text{crit}} = \frac{\pi}{\sqrt{2}} \approx 2.221, \quad \text{Ra}_{\text{crit}} = \frac{27\pi^4}{4} \approx 657.511, \quad (3.10)$$

see, for example, [1].

The main physical conclusion yielded by the subsequent results is that the depth dependence of material properties stabilize the conductive state (the critical Rayleigh number increases) but the critical wavenumbers remain almost unchanged. However, the incorporation of adiabatic heating/cooling and internal heating results in a *substantial* stabilization (the critical Rayleigh number increases by several orders of magnitude) and in relatively high values of the critical wavenumbers.

3.2.1. Depth-dependent thermal conductivity

Here, we assume that k is given by (2.3a) while α and μ are constant. Numerical solution of (3.8) then yields the critical values of wavenumber a and Rayleigh number Ra which are summarized in Table 1 for different values of the dissipation number Di and Rayleigh number for heat sources Ra_q .

3.2.2. Depth-dependent thermal expansivity

Here, we assume that α is given by (2.3b) while k and μ are constant. Numerical solution of (3.8) then yields the critical values of wavenumber a and Rayleigh number Ra which are summarized in Table 2 for different values of the dissipation number Di and Rayleigh number for heat sources Ra_q .

Table 1Critical pairs a_{crit} , Ra_{crit} for depth-dependent thermal conductivity.

b_k	Di	Ra_q	a_{crit}	Ra_{crit}
1	0	0	2.221	976.933
5	0	0	2.220	2 210.40
10	0	0	2.218	3 767.83
1	0.5	0	6.334	404 688
5	0.5	0	7.086	344 453
10	0.5	0	7.155	342 840
1	0.5	10^6	7.565	225 133
5	0.5	10^6	7.612	274 150
10	0.5	10^6	7.465	303 949

Table 2Critical pairs a_{crit} , Ra_{crit} for depth-dependent thermal expansivity.

b_α	Di	Ra_q	a_{crit}	Ra_{crit}
0.5	0	0	2.222	840.560
1.0	0	0	2.223	1 065.19
2.0	0	0	2.227	1 666.83
0.5	0.5	0	3.639	387 573
1.0	0.5	0	4.294	258 856
2.0	0.5	0	3.854	138 156
0.5	0.5	10^6	7.068	139 911
1.0	0.5	10^6	6.694	103 769
2.0	0.5	10^6	6.067	39 607.8

Table 3Critical pairs a_{crit} , Ra_{crit} for depth-dependent dynamic viscosity.

b_μ	Di	Ra_q	a_{crit}	Ra_{crit}
1.0	0	0	2.199	1 113.68
2.5	0	0	2.093	2 679.48
4.5	0	0	1.921	9 520.92
1.0	0.5	0	5.225	478 900
2.5	0.5	0	4.961	485 309
4.5	0.5	0	4.479	498 954
1.0	0.5	10^6	7.325	186 361
2.5	0.5	10^6	7.080	201 026
4.5	0.5	10^6	6.645	227 321

3.2.3. Depth-dependent dynamic viscosity

Here, we assume that μ is given by (2.3c) while k and α are constant. Numerical solution of (3.8) then yields the critical values of wavenumber a and Rayleigh number Ra which are summarized in Table 3 for different values of the dissipation number Di and Rayleigh number for heat sources Ra_q .

4. Convection patterns in slightly supercritical regime

In order to study the qualitative character of the convection in the slightly supercritical regime, we restrict ourselves to a two-dimensional setting. Since we portray the convection cells in this section, recall that the vertical axis z of the coordinate system is pointing downwards, that is $z = 0$ at the surface and $z = 1$ at the bottom of the convecting layer. In addition, we emphasize that the periodic cell Ω now simplifies to

$$\Omega = \left(0, \frac{2\pi}{a}\right) \times (0, 1), \quad (4.1)$$

where a represents the wavenumber corresponding to the x coordinate. Since we are interested in the slightly supercritical behaviour of the convection we prescribe the wavenumber to match the corresponding critical value a_{crit} obtained in the previous section.

The dynamics in the slightly supercritical regime is investigated using the weakly nonlinear analysis. In this respect we follow the approach by [18] and [16], while we make modifications that allow us to work within the setting wherein the corresponding linearized operator is not normal. The non-normality of the corresponding operator stems from the fact

that we work with depth-dependent coefficients, and this feature makes the problem different from the weakly nonlinear analysis of the classical thermal convection problem.

4.1. Perturbation equations in abstract form

Denoting $\psi =_{\text{def}} [\vartheta, \mathbf{v}]^\top$ and introducing the function space

$$H =_{\text{def}} \{\psi \in [W^{1,2}(\Omega)]^3 \mid \text{div} \mathbf{v} = 0\}, \quad (4.2)$$

where $W^{1,2}(\Omega)$ denotes the standard Sobolev function space, we can rewrite the system of equations (2.12) governing the evolution of the perturbations $[\vartheta, \mathbf{v}, p]$ in the form

$$\frac{\partial \psi}{\partial t} = \mathcal{L}(\psi) - \text{Pr} \delta p + \mathcal{N}(\psi, \psi), \quad (4.3)$$

where we have defined the linear operator $\mathcal{L} : H \rightarrow H$

$$\mathcal{L} =_{\text{def}} \begin{bmatrix} k\Delta + \frac{dk}{dz} \frac{\partial}{\partial z} & 0 & -\frac{d\theta_0}{dz} + \text{Di}\alpha \left(\theta_0 + \frac{\Xi}{\text{Ra}} \right) \\ 0 & \text{Pr}\mu\Delta + \text{Pr} \frac{d\mu}{dz} \frac{\partial}{\partial z} & \text{Pr} \frac{d\mu}{dz} \frac{\partial}{\partial x} \\ -\text{PrRa}\alpha & 0 & \text{Pr}\mu\Delta + 2\text{Pr} \frac{d\mu}{dz} \frac{\partial}{\partial z} \end{bmatrix}, \quad (4.4)$$

the quadratic nonlinear operator $\mathcal{N} : H \times H \rightarrow H$

$$\mathcal{N}(\phi, \psi) = \mathcal{N} \left(\begin{bmatrix} \vartheta_\phi \\ \mathbf{v}_\phi^x \\ \mathbf{v}_\phi^z \end{bmatrix}, \begin{bmatrix} \vartheta_\psi \\ \mathbf{v}_\psi^x \\ \mathbf{v}_\psi^z \end{bmatrix} \right) =_{\text{def}} \begin{bmatrix} -(\mathbf{v}_\phi \cdot \nabla) \vartheta_\psi + \text{Di}\alpha \mathbf{v}_\phi^z \vartheta_\psi + \frac{\text{Di}}{\text{Ra}} \mu \mathbb{D}_\phi : \mathbb{D}_\psi \\ -(\mathbf{v}_\phi \cdot \nabla) \mathbf{v}_\psi^x \\ -(\mathbf{v}_\phi \cdot \nabla) \mathbf{v}_\psi^z \end{bmatrix}, \quad (4.5)$$

and we have also introduced the operator $\delta =_{\text{def}} [0, \nabla]^\top$.

Let us define the inner product on the function space H by the formula

$$\langle \phi, \psi \rangle =_{\text{def}} \int_0^1 \int_0^{\frac{2\pi}{a}} \phi(x, z) \bullet \overline{\mathbb{M}\psi(x, z)} \, dx dz, \quad (4.6)$$

where $\phi, \psi \in H$, and

$$\mathbb{M} =_{\text{def}} \begin{bmatrix} \text{Ra} & 0 & 0 \\ 0 & \text{Pr}^{-1} & 0 \\ 0 & 0 & \text{Pr}^{-1} \end{bmatrix}, \quad (4.7)$$

4.2. Completeness of eigenfunctions

For the classical Boussinesq approximation ($k, \alpha, \mu = 0, \text{Di} = 0, \text{Ra}_q = 0$), the linear operator $\mathcal{L}$ can be shown to be *normal* (self-adjoint even) under the inner product (4.6). In this special case, the eigenfunctions of the linear eigenvalue problem

$$\mathcal{L}(\psi^{(i)}) = \sigma^{(i)} \psi^{(i)}, \quad (4.8)$$

where $\sigma^{(i)}$ denotes the i -th eigenvalue and $\psi^{(i)}$ the corresponding eigenfunction, form an orthogonal and complete set in H . (Regarding the rigorous analysis of completeness of eigenspaces of operators met in hydrodynamic stability we refer the reader to [28].) However, in the general case (4.4) the normality property is lost – even under simplifications such as $k = k(z), \alpha = 1, \mu = 1$ there does not exist an inner product which would make the operator $\mathcal{L}$ normal (let alone self-adjoint). The numerical results show that for Prandtl numbers sufficiently large (greater than approximately 10^3 which is indeed our case as we set $\text{Pr} = 10^9$) the eigenvalues of (4.8) are *real and distinct*. This leads us to the assumption that even in the case of the extended Boussinesq approximation the eigenfunctions $\{\psi^{(i)}\}_{i=1}^{+\infty}$ form a complete set in H , that is an arbitrary function $\psi \in H$ can be expanded in terms of $\psi^{(i)}$.

Since the numerical evidence suggests that the eigenvalues $\{\sigma^{(i)}\}_{i=1}^{+\infty}$ are real and distinct, let them be ordered in a descending manner

$$\sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)} > \dots \quad (4.9)$$

Note that when Ra attains the critical value, we obtain $\sigma^{(1)} = 0$, and $\sigma^{(i)} < 0$ for $i \geq 2$.

4.3. Fourier series expansion

Since we assume x -periodicity we can expand the eigenfunctions of (4.8) into their Fourier series

$$\psi^{(i)}(x, z) = \sum_{n=-\infty}^{+\infty} \psi_n^{(i)}(z) \exp(inax). \quad (4.10)$$

Functions $\psi_n^{(i)}$ belong to the space $\tilde{H} =_{\text{def}} \{\tilde{\psi} \in [W^{1,2}((0, 1))]^3 \mid ina\tilde{\psi}^x + \frac{d\tilde{\psi}^z}{dz} = 0\}$ parameterized by n and a , and satisfy

$$\mathcal{L}_n(\psi_n^{(i)}) = \sigma_n^{(i)} \psi_n^{(i)}, \quad (4.11)$$

where $\mathcal{L}_n$ is the Fourier transformed version of $\mathcal{L}$, i.e.

$$\mathcal{L}_n =_{\text{def}} \mathcal{L} \Big|_{\frac{\partial}{\partial x} \rightarrow ina, \frac{\partial}{\partial z} \rightarrow \frac{d}{dz}}. \quad (4.12)$$

Analogously, as in Section 3.1, the eigenvalue problem (4.11) can be reformulated into an eigenvalue problem for the perturbation $[\vartheta, \tilde{\psi}^z]$ only. (Here, we use the incompressibility condition (2.1a).) This way we, in fact, arrive at the eigenvalue problem (3.5a) with a replaced by na . For further use we also define the inner product on $\tilde{H}$ by the formula

$$\langle \tilde{\phi}, \tilde{\psi} \rangle =_{\text{def}} \int_0^1 \tilde{\phi}(z) \bullet \overline{\mathbb{M} \tilde{\psi}(z)} dz. \quad (4.13)$$

4.4. Derivation of amplitude equation

Under the assumption of the completeness of the eigenfunctions of (4.8) and considering its Fourier expansions, we can expand any function ψ into

$$\psi(t, x, z) = \sum_{i=1}^{+\infty} \sum_{n=-\infty}^{+\infty} A_n^{(i)}(t) \psi_n^{(i)}(z) \exp(inax). \quad (4.14)$$

We require that ψ is a real function, hence $A_{-n}^{(i)} = \overline{A_n^{(i)}}$, and $\psi_{-n}^{(i)} = \overline{\psi_n^{(i)}}$. Furthermore, it can be shown that in fact $A_n^{(i)} \in \mathbb{R}$. Substituting (4.14) into (4.3) and taking the inner product $\langle \cdot, \cdot \rangle$ of the resulting equation with $\psi_l^{(k)} \exp(ilax)$, where $k \in \mathbb{N}$ and $l \in \mathbb{Z}$, yields

$$\sum_{i=1}^{+\infty} \frac{dA_l^{(i)}}{dt} \langle \psi_l^{(i)}, \psi_l^{(k)} \rangle = \sum_{i=1}^{+\infty} \sigma_l^{(i)} A_l^{(i)} \langle \psi_l^{(i)}, \psi_l^{(k)} \rangle + \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \sum_{n=-\infty}^{+\infty} A_n^{(i)} A_{l-n}^{(j)} \langle \tilde{\mathcal{N}}(\psi_n^{(i)}, \psi_{l-n}^{(j)}), \psi_l^{(k)} \rangle, \quad (4.15)$$

where $\tilde{\mathcal{N}}$ is the Fourier transformed version of $\mathcal{N}$, that is

$$\tilde{\mathcal{N}}(\psi_n^{(i)}, \psi_m^{(j)}) =_{\text{def}} \mathcal{N}(\psi_n^{(i)}, \psi_m^{(j)}) \Big|_{\frac{\partial}{\partial x} \psi_n^{(i)} \rightarrow ina \psi_n^{(i)}, \frac{\partial}{\partial x} \psi_m^{(j)} \rightarrow ina \psi_m^{(j)}, \frac{\partial}{\partial z} \rightarrow \frac{d}{dz}} \quad (4.16)$$

To derive the amplitude equation we now restrict ourselves to a finite number of eigenfunctions, l_0 say. Equation (4.15) then reduces to

$$\mathbb{P}_l \frac{d\mathbf{A}_l}{dt} = \mathbb{P}_l \mathbb{D}_l \mathbf{A}_l + \sum_{i=1}^{l_0} \sum_{j=1}^{l_0} \sum_{n=-\infty}^{+\infty} A_n^{(i)} A_{l-n}^{(j)} \mathbf{N}_{n,l-n}^{i,j} \quad (4.17)$$

where we have introduced a matrix $\mathbb{P}_l$, a diagonal matrix $\mathbb{D}_l$, and vectors $\mathbf{A}_l$, $\mathbf{N}_{n,l-n}^{i,j}$ by the following formulae

$$[\mathbb{P}_l]_{ki} =_{\text{def}} \langle \psi_l^{(i)}, \psi_l^{(k)} \rangle, \quad (4.18a)$$

$$[\mathbb{D}_l]_{ii} =_{\text{def}} \sigma_l^{(i)}, \quad (4.18b)$$

$$[\mathbf{A}_l]_k =_{\text{def}} A_l^{(i)}, \quad (4.18c)$$

$$[\mathbf{N}_{n,l-n}^{i,j}]_k =_{\text{def}} \langle \tilde{\mathcal{N}}(\psi_n^{(i)}, \psi_{l-n}^{(j)}), \psi_l^{(k)} \rangle. \quad (4.18d)$$

Since we assumed linear independence of the eigenfunctions, the matrix $\mathbb{P}_l$ is invertible (the subsequent numerical computations indeed confirm that $\mathbb{P}_l$ is numerically far from singular). Multiplying (4.17) by $\mathbb{P}_l^{-1}$ from the left then yields

$$\frac{dA_l^{(k)}}{dt} = \sigma_l^{(k)} A_l^{(k)} + \sum_{i=1}^{l_0} \sum_{j=1}^{l_0} \sum_{n=-\infty}^{+\infty} \lambda_{n,l-n}^{i,j,k} A_n^{(i)} A_{l-n}^{(j)}, \quad (4.19)$$

where we have defined

$$\lambda_{n,l-n}^{i,j,k} =_{\text{def}} [\mathbb{P}_l^{-1} \mathbf{N}_{n,l-n}^{i,j}]_k. \quad (4.20)$$

Equation (4.19) represents an infinite-dimensional system of ordinary differential equations for the quotients $A_l^{(k)}$. However, we are interested in the evolution equation for the amplitude of the first unstable mode, that is in the evolution equation for the term $A_1^{(1)}$. Using the notation

$$A =_{\text{def}} A_1^{(1)}, \quad (4.21)$$

Table 4

Convergence of Λ_1 in terms of I_0 for $k(z) = 1 + 5z$, $\alpha(z) = 1$, $\mu(z) = 1$, $\text{Pr} = 10^9$, $\text{Ra} = 280\,000$, $\text{Di} = 0.5$, $\text{Ra}_q = 10^6$, $a = a_{\text{crit}} = 7.612$.

I_0	Λ_1
2	-1.18219303
4	-2.32396270
8	-3.52020696
12	-3.44083661
20	-3.44074136
40	-3.44065954

we are then concerned with the equation

$$\frac{dA}{dt} = \sigma_1^{(1)} A + \sum_{i=1}^{I_0} \sum_{j=1}^{I_0} \sum_{n=-\infty}^{+\infty} \lambda_{n,1-n}^{i,j,1} A_n^{(i)} A_{1-n}^{(j)}. \quad (4.22)$$

Following the methodology of [16] we can approximate the nonlinear term in (4.22) with the use of the evolution equation (4.19) for the stable modes. The evolution equation for the unstable mode A then simplifies to

$$\frac{dA}{dt} = \sigma_1^{(1)} A + \Lambda_1 A^3, \quad (4.23)$$

where we have denoted

$$\Lambda_1 =_{\text{def}} \sum_{i=1}^{I_0} -\frac{2}{\sigma_0^{(i)}} \lambda_{1,-1}^{i,1,i} (\lambda_{0,1}^{i,1,1} + \lambda_{1,0}^{i,1,1}) + \sum_{i=1}^{I_0} -\frac{1}{\sigma_2^{(i)}} \lambda_{1,-1}^{i,1,i} (\lambda_{2,-1}^{i,1,1} + \lambda_{-1,2}^{i,1,1}). \quad (4.24)$$

In the literature, equation (4.23) is often called the *amplitude equation*. As can be seen from the definition of Λ_1 (4.24), the amplitude equation is well-defined and applicable only in the interval of supercritical values of Rayleigh number for which the eigenvalues $\sigma_0^{(i)}$, $\sigma_2^{(i)}$ remain negative.

4.5. Numerical evaluation of amplitude equation

Table 4 shows the convergence of the coefficient Λ_1 in terms of the truncation level I_0 for a setting specified in its caption. The Rayleigh number is chosen such that it lies above the critical value ($\text{Ra}_{\text{crit}} = 274\,150$) while $\sigma_1^{(1)}$ still remains the only positive eigenvalue. We see that the coefficient Λ_1 is negative and thus for this choice of parameters the amplitude equation has two nonnegative stable solutions, $A \approx \pm 2.656$.

Convergence results similar to the one in Table 4 can be obtained for all the cases considered further. Moreover, the coefficient Λ_1 always stays negative in the applicable range of supercritical Rayleigh numbers. This implies that the amplitude equation (4.23) leads to the *pitchfork bifurcation* at the critical value of Rayleigh number.

In the following subsections we compute the stable solutions of the amplitude equation (4.23) for constant material parameters except for a single depth-dependent one. In each case we evaluate the modification of the pitchfork bifurcation compared to the classical case; for this reason we set $\text{Di} = 0$, $\text{Ra}_q = 0$ to ensure proximity to the classical diagram.

It is of interest to also plot the so-called Nusselt number which measures the ratio of convective to conductive heat transfer at the top boundary. In particular, we compute the *horizontally averaged* Nusselt number as in [29]. The horizontally averaged Nusselt number is defined as

$$\text{Nu}_{\text{avg}} =_{\text{def}} \frac{\int_{x=-\frac{\pi}{a}}^{\frac{\pi}{a}} \nabla \theta \cdot \mathbf{n}|_{z=0} dx}{\int_{x=-\frac{\pi}{a}}^{\frac{\pi}{a}} \nabla \theta_0 \cdot \mathbf{n}|_{z=0} dx}. \quad (4.25)$$

Expanding the formula (4.14) for the perturbation ϑ yields the following approximation for the perturbation ϑ^3

$$\vartheta \approx A_0^{(1)} \vartheta_0^{(1)}(z) + 2A_1^{(1)} \vartheta_1^{(1)}(z) \cos(ax) + A_0^{(2)} \vartheta_0^{(2)}(z), \quad (4.26)$$

where the terms $A_0^{(1)}$ and $A_0^{(2)}$ can be approximated by the formula

$$A_0^{(k)} \approx -\frac{2}{\sigma_0^{(k)}} \lambda_{1,-1}^{1,1,k} A^2, \quad (4.27)$$

³ It is necessary to keep the term $A_0^{(2)}$ as in the case of the classical Boussinesq approximation the term $A_0^{(1)}$ vanishes and the horizontally averaged Nusselt number would be equal to one for all values of the Rayleigh number.

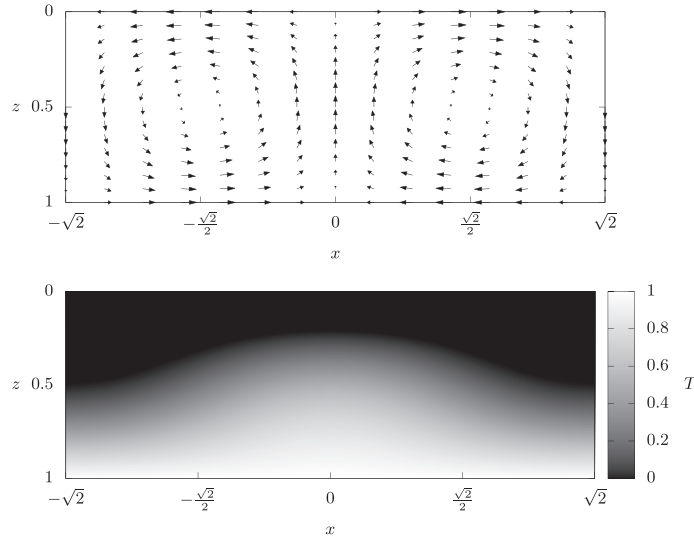


Fig. 3. Velocity and temperature fields in convection cell for $k(z) = 1$, $\alpha(z) = 1$, $\mu(z) = 1$, $\text{Pr} = 10^9$, $\text{Ra} = 1000$, $\text{Di} = 0$, $\text{Ra}_q = 0$.

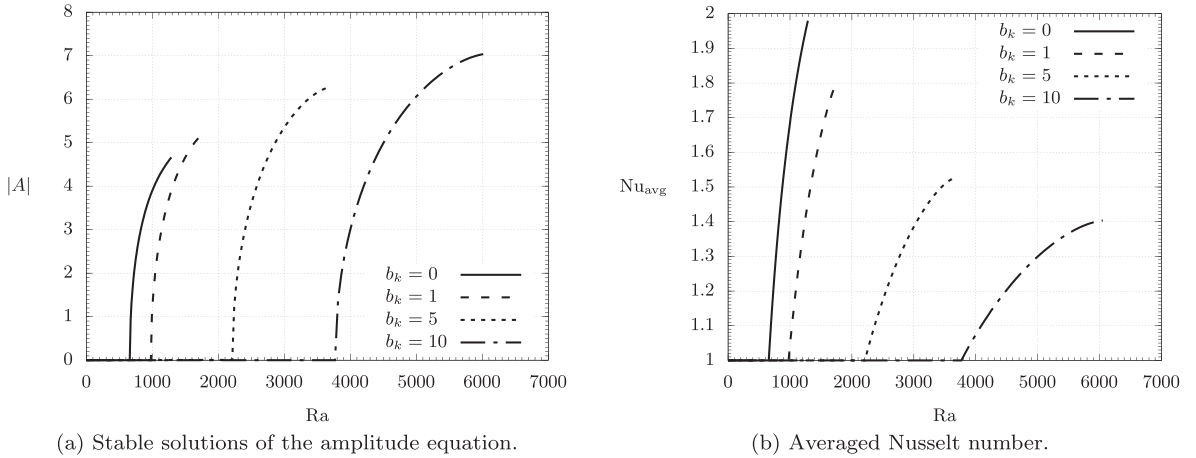


Fig. 4. Supercritical convective regime for parameters $k(z) = 1 + b_k z$, $\alpha(z) = 1$, $\mu(z) = 1$, $\text{Pr}_s = 10^9$, $\text{Di} = 0$, $\text{Ra}_q = 0$.

which is a consequence of the relation (4.19). Using the approximation (4.26) together with the fact that $\theta = \theta_0 + \vartheta$ then yields

$$\text{Nu}_{\text{avg}} \approx 1 + \frac{A_0^{(1)} \frac{d\vartheta_0^{(1)}}{dz} \Big|_{z=0} + A_0^{(2)} \frac{d\vartheta_0^{(2)}}{dz} \Big|_{z=0}}{\frac{d\theta_0}{dz} \Big|_{z=0}}. \quad (4.28)$$

Note that the second term on the right-hand side of (4.26) has no effect in the explicit formula for the averaged Nusselt number since the averaging makes it vanish.

Finally, we portray the convection cells for the general case $\text{Di} \neq 0$, $\text{Ra}_q \neq 0$ to emphasize the difference of the qualitative character of convection compared to the classical case. The classical Boussinesq approximation yields a convection cell with velocity and temperature fields depicted in Fig. 3 where the values $\text{Pr} = 10^9$, $\text{Ra} = 1000$ give $A \approx \pm 3.900$. In order to decrease computational time we use the truncation level $l_0 = 12$ in all the subsequent computations.

4.5.1. Depth-dependent thermal conductivity

Here, we assume that k is given by (2.3a) while α and μ are constant. Fig. 4 shows the stable solutions of the amplitude equation (4.23) and the Nusselt number as functions of the Rayleigh number for various values of b_k while $\text{Di} = 0$, $\text{Ra}_q = 0$. The classical case corresponds to the case $b_k = 0$.

In Fig. 5 we depict the convection cell with its velocity and temperature fields of the general case including the effects of dissipation, adiabatic heating/cooling and heat sources.

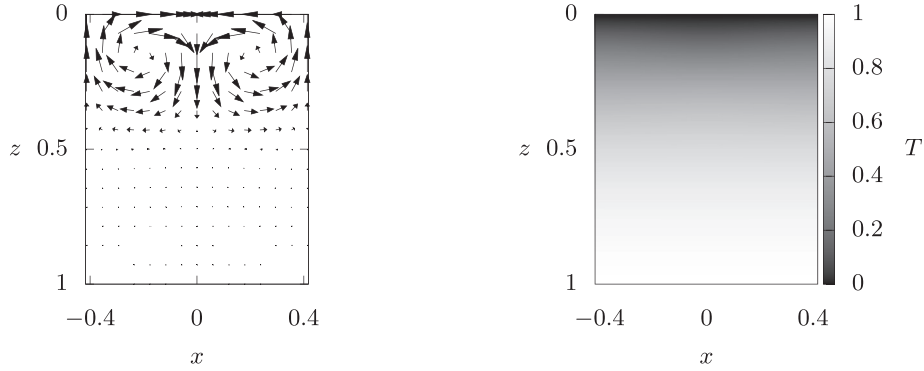


Fig. 5. Convection cell for $k(z) = 1 + 5z$, $\alpha(z) = 1$, $\mu(z) = 1$, $\text{Pr} = 10^9$, $\text{Ra} = 280\,000$, $\text{Di} = 0.5$, $\text{Ra}_q = 10^6$.

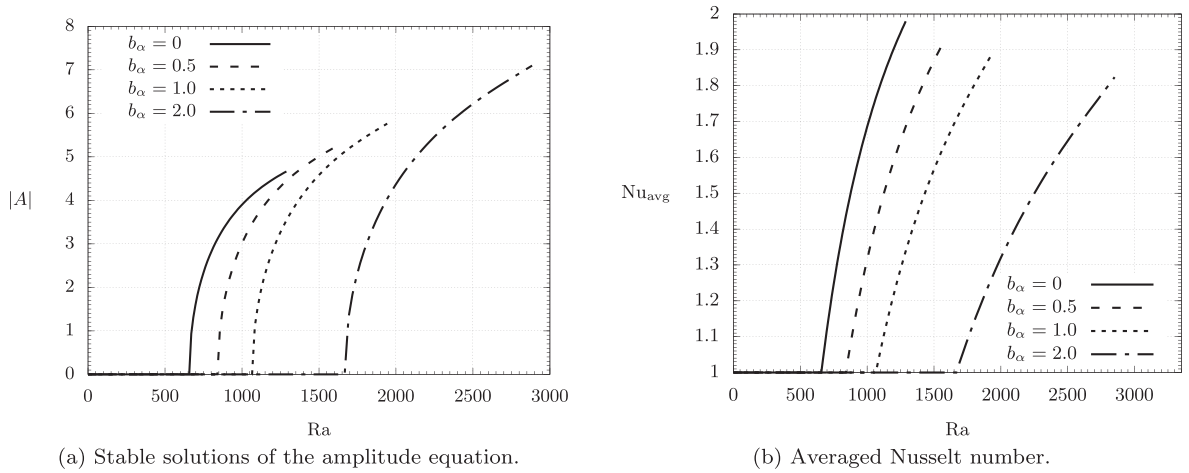


Fig. 6. Supercritical convective regime for parameters $k(z) = 1$, $\alpha(z) = \exp(-b_\alpha z)$, $\mu(z) = 1$, $\text{Pr} = 10^9$, $\text{Di} = 0$, $\text{Ra}_q = 0$.

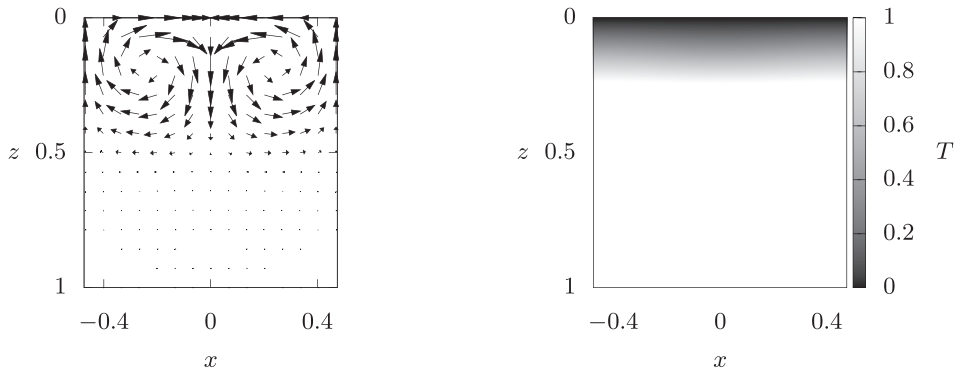


Fig. 7. Convection cell for $k(z) = 1$, $\alpha(z) = \exp(-z)$, $\mu(z) = 1$, $\text{Pr} = 10^9$, $\text{Ra} = 120\,000$, $\text{Di} = 0.5$, $\text{Ra}_q = 10^6$.

4.5.2. Depth-dependent thermal expansivity

Here, we assume that α is given by (2.3b) while k and μ are constant. Fig. 6 shows the stable solutions of the amplitude equation (4.23) and the Nusselt number as functions of the Rayleigh number for various values of b_α while $\text{Di} = 0$, $\text{Ra}_q = 0$. The classical case corresponds to the case $b_\alpha = 0$.

In Fig. 7 we depict the convection cell with its velocity and temperature fields of the general case including the effects of dissipation, adiabatic heating/cooling and heat sources.

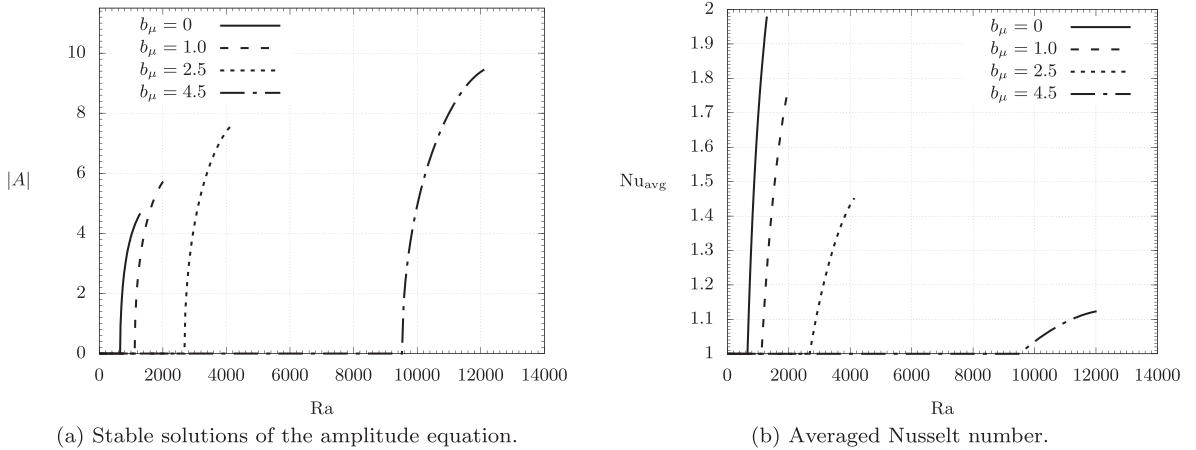


Fig. 8. Supercritical convective regime for parameters $k(z) = 1$, $\alpha(z) = 1$, $\mu(z) = \exp(b_\mu z)$, $Pr = 10^9$, $Di = 0$, $Ra_q = 0$.

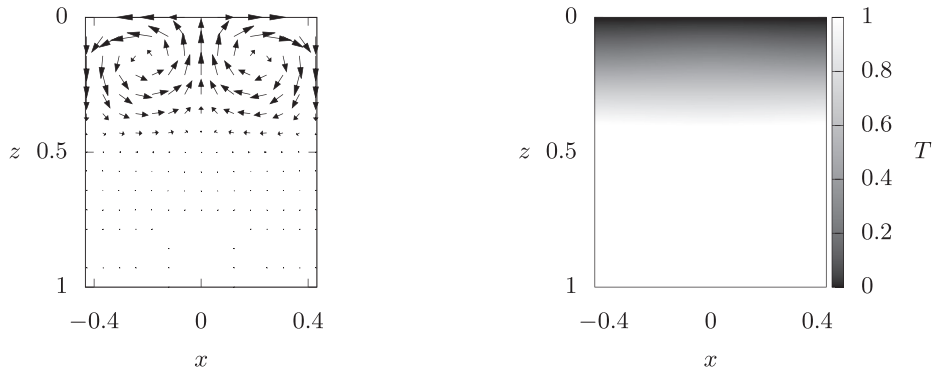


Fig. 9. Convection cell for $k(z) = 1$, $\alpha(z) = 1$, $\mu(z) = \exp(z)$, $Pr = 10^9$, $Ra = 200\,000$, $Di = 0.5$, $Ra_q = 10^6$.

4.5.3. Depth-dependent dynamic viscosity

Here, we assume that μ is given by (2.3c) while k and α are constant. Fig. 8 shows the stable solutions of the amplitude equation (4.23) and the Nusselt number as functions of the Rayleigh number for various values of b_μ while $Di = 0$, $Ra_q = 0$. The classical case corresponds to the case $b_\mu = 0$.

In Fig. 9 we depict the convection cell with its velocity and temperature fields of the general case including the effects of dissipation, adiabatic heating/cooling and heat sources.

5. Conclusion

The standard approach to Rayleigh–Bénard problem assumes constant material parameters and does not take into account dissipation, adiabatic heating/cooling nor heat sources. The standard procedures examining the onset of convection and the supercritical regime of thermal convection have thus been generalized for extended Boussinesq approximation, which includes these effects. The material parameters used in our study are those approximately corresponding to the properties of the Earth's mantle.

Following the theory of linear stability analysis we have linearized the governing equations in the vicinity of the stationary reference state, and we have computed the spectrum of the corresponding linear operator. We have found the critical wavenumbers and critical Rayleigh numbers for various combinations of the depth-dependent material coefficients with the effects of dissipation, adiabatic cooling/heating and heat sources. The main difference from the classical case is caused by the presence of adiabatic heating/cooling, which significantly raises the threshold for the onset of convection, see Section 4.5 for the numerical values.

We have further investigated the qualitative characteristics of thermal convection for slightly supercritical values of Rayleigh number in a two-dimensional setting. Using the weakly nonlinear analysis we have derived the amplitude equation for the extended Boussinesq approximation, and we have consequently identified the dominant velocity and temperature fields in the supercritical regimes for various combinations of the depth-dependent material coefficients with the effects of dissipation, adiabatic cooling/heating and heat sources. The dominating convection patterns have been found to substan-

tially differ from the standard ones, see [Section 4.5](#) for details. In most cases the velocity field in the convective regime is localised close to the top (cold) boundary, while the fluid is almost quiescent near the hot (bottom) boundary.

References

- [1] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.
- [2] C. Matyska, D.A. Yuen, Lower mantle material properties and convection models of multiscale plumes, *Geological Society of America Special Papers* (2007) 137–163.
- [3] K. Rajagopal, G. Saccomandi, L. Vergori, Stability analysis of the rayleigh–bénard convection for a fluid with temperature- and pressure-dependent viscosity, *Z. angew. Math. Phys.* 60 (4) (2009) 739–755, doi:[10.1007/s00033-008-8062-6](#).
- [4] K.R. Rajagopal, G. Saccomandi, L. Vergori, On the oberbeck-boussinesq approximation for fluids with pressure dependent viscosities, *Nonlinear Anal. Real World Appl.* 10 (2) (2009) 1139–1150.
- [5] S. Rosenblat, Thermal convection in a viscoelastic liquid, *J. Non-Newtonian Fluid Mech.* 21 (2) (1986) 201–223, doi:[10.1016/0377-0257\(86\)80036-2](#).
- [6] H. Demir, Rayleigh–benard convection of viscoelastic fluid, *Appl. Math. Comput.* 136 (2) (2003) 251–267, doi:[10.1016/S0096-3003\(02\)00036-X](#).
- [7] H. Demir, F. Akylidiz, Unsteady thermal convection of a non-newtonian fluid, *Int. J. Eng. Sci.* 38 (17) (2000) 1923–1938, doi:[10.1016/S0020-7225\(00\)00011-2](#).
- [8] M. Bouteraa, C. Nouar, E. Plaut, C. Metivier, A. Kalck, Weakly nonlinear analysis of rayleigh–bénard convection in shear-thinning fluids: nature of the bifurcation and pattern selection, *J. Fluid Mech.* 767 (2015) 696734, doi:[10.1017/jfm.2015.64](#).
- [9] M. Bouteraa, C. Nouar, Rayleigh–bénard convection in non-newtonian carreau fluids with arbitrary conducting boundaries, *Int. Commun. Heat Mass* 76 (2016) 77–84.
- [10] M. Bouteraa, C. Nouar, Weakly nonlinear analysis of rayleigh–bénard convection in a non-newtonian fluid between plates of finite conductivity: influence of shear-thinning effects, *Phys. Rev. E* 92 (2015) 063017, doi:[10.1103/PhysRevE.92.063017](#).
- [11] G. Gupta, M. Massoudi, Flow of a generalized second grade fluid between heated plates, *Acta Mech.* 99 (1–4) (1993) 21–33, doi:[10.1007/BF01177232](#).
- [12] B. Straughan, Energy stability in the benard problem for a fluid of second grade, *Z. angew. Math. Phys.* 34 (4) (1983) 502–509, doi:[10.1007/BF00944711](#).
- [13] F. Franchi, B. Straughan, Convection, stability and uniqueness for a fluid of third grade, *Int. J. Non-Linear Mech.* 23 (5) (1988) 377–384, doi:[10.1016/0020-7462\(88\)90035-2](#).
- [14] M. Massoudi, A. Vaidya, R. Wulandana, Natural convection flow of a generalized second grade fluid between two vertical walls, *Nonlinear Anal. Real World Appl.* 9 (1) (2008) 80–93.
- [15] M.S. Aghighi, A. Ammar, C. Metivier, M. Gharagozlu, Rayleigh–bénard convection of casson fluids, *Int. J. Therm. Sci.* 127 (2018) 79–90, doi:[10.1016/j.ijthermalsci.2018.01.016](#).
- [16] M.C. Cross, Derivation of the amplitude equation at the rayleigh–bénard instability, *Phys. Fluids* 23 (9) (1980) 1727–1731, doi:[10.1063/1.863198](#).
- [17] K. Fujimura, Methods of centre manifold and multiple scales in the theory of weakly nonlinear stability for fluid motions, *Proc. R. Soc. A: Math. Phys. Eng. Sci.* 434 (1892) (1991) 719–733, doi:[10.1098/rspa.1991.0124](#).
- [18] K. Fujimura, Centre manifold reduction and the stuart-landau equation for fluid motions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 453 (1997) 181–203, doi:[10.1098/rspa.1997.0011](#).
- [19] Y. Kagei, M. Růžička, G. Thäter, Natural convection with dissipative heating, *Commun. Math. Phys.* 214 (2000) 287–313, doi:[10.1007/s002200000275](#).
- [20] K.R. Rajagopal, M. Růžička, A.R. Srinivasa, On the oberbeck–boussinesq approximation, *Math. Models Methods Appl. Sci.* 6 (8) (1996) 1157–1167.
- [21] R. Zeytounian, Joseph boussinesq and his approximation: a contemporary view, *C. R. Mec.* 331 (8) (2003) 575–586, doi:[10.1016/S1631-0721\(03\)00120-7](#).
- [22] G. Schubert, D.L. Turcotte, P. Olsen, *Mantle convection in the earth and planets*, Cambridge Monographs on Mechanics, Cambridge University Press, 2001.
- [23] C. Barus, Isotherms, isopiestic and isometrics relative to viscosity, *Amer. J. Sci.* 45 (1893) 87–96.
- [24] S. Bair, P. Kottke, Pressure-viscosity relationships for elastohydrodynamics, *Tribol. Trans.* 46 (3) (2003) 289–295, doi:[10.1080/10402000308982628](#).
- [25] L.N. Trefethen, *Spectral methods in MATLAB*, Software, Environments, and Tools, 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
- [26] J.P. Boyd, *Chebyshev and fourier spectral methods*, 2nd, Dover Publications Inc., Mineola, NY, 2001.
- [27] J.A. Weideman, S.C. Reddy, A MATLAB differentiation matrix suite, *ACM Trans. Math. Softw.* 26 (4) (2000) 465–519, doi:[10.1145/365723.365727](#).
- [28] R.C. Di Prima, G.J. Habetler, A completeness theorem for non-selfadjoint eigenvalue problems in hydrodynamic stability, *Arch. Ration. Mech. Anal.* 34 (3) (1969) 218–227, doi:[10.1007/BF00281139](#).
- [29] B. Bhadauria, P. Siddheshwar, J. Kumar, O. Suthar, Weakly nonlinear stability analysis of temperature/gravity-modulated stationary rayleigh–bñard convection in a rotating porous medium, *Transp. Porous. Med.* 92 (2011) 633647.