

Matrix eigenvalue method for free-oscillations modelling of spherical elastic bodies

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SUMMARY

Deformations and changes of the gravitational potential of pre-stressed self-gravitating elastic bodies caused by free oscillations are described by means of the momentum and Poisson equations and the constitutive relation. For spherically symmetric bodies, the equations and boundary conditions are transformed into ordinary differential equations of the second order by the spherical harmonic decomposition and further discretized by highly accurate pseudospectral difference schemes on Chebyshev grids; we pay special attention to the conditions at the centre of the models. We thus obtain a series of matrix eigenvalue problems for eigenfrequencies and eigenfunctions of the free oscillations. Accuracy of the presented numerical approach is tested by means of the Rayleigh quotients calculated for the eigenfrequencies up to 500 mHz. Both the modal frequencies and eigenfunctions are benchmarked against the output from the Mineos software package based on shooting methods. The presented technique is a promising alternative to widely used methods because it is stable and with a good capability up to high frequencies.

Key words: Numerical solutions; Surface waves and free oscillations.

1 INTRODUCTION

Normal modes or free oscillations are components of standing waves with an infinite number of degrees of freedom that are created from the constructive interference of body and surface waves circling the Earth multiple times. For the commonly used spherically symmetric, non-rotating, perfectly elastic and isotropic (SNREI) Earth models the oscillations are composed of two distinct types of modes. The spheroidal modes correspond to standing Rayleigh waves and change the external shape and internal density of the body, and the toroidal modes correspond to standing Love waves with purely tangential displacements and leave the shape and the density distribution unaffected.

The theoretical description of this phenomenon for simple models appeared more than a century ago; for example, the normal modes of a self-gravitating homogeneous solid sphere were discussed by Love (1911) and more realistic Earth models were treated, for example, by Pekeris & Jarosch (1958) and Alterman *et al.* (1959). Nevertheless, the complete correct theory was first published much later by Woodhouse & Dahlen (1978), who followed Dahlen (1972) and derived linearized equations of motion and linearized boundary and continuity conditions governing small elastic-gravitational disturbances away from an equilibrium state of an arbitrary, uniformly rotating, self-gravitating, perfectly elastic Earth model with an arbitrary initial static stress field.

The standard approach of computing eigenfrequencies and eigenfunctions of spheroidal (toroidal) oscillations of the SNREI model is based on shooting methods, where the numerical integration of radial six (two) ordinary differential equations of the first order is performed at a given frequency, with the eigenfrequencies being defined by the vanishing of an appropriate function at the surface (e.g. Takeuchi & Saito 1972; Woodhouse 1988; Dahlen & Tromp 1998). Two examples of computer codes based on such a procedure are John Woodhouse's Obani and Guy Masters' Mineos that are the successors of an earlier program written by Freeman Gilbert. In this study, we use as reference the Mineos software package available online at <http://geodynamics.org/cig/software/mineos>. An alternative approach is based on a Rayleigh–Ritz variational procedure, where a Lagrangian integral is minimized numerically on a set of appropriate basis functions (e.g. Wiggins 1976; Buland & Gilbert 1984; Dahlen & Tromp 1998).

Instead of these methods, we have developed a different approach where the direct numerical discretization of the ordinary differential equations by high-accuracy pseudospectral schemes (Fornberg 1996) leads us to a series of matrix eigenvalue problems for evaluating eigenfrequencies and eigenfunctions. Basically, these problems correspond to the matrix eigenvalue problems derived by the Rayleigh–Ritz

variational procedure but the numerical discretization is performed in a completely different way and thus the extents of numerically obtained spectra (corresponding to matrix dimensions), accuracy of eigenvalues and discretization of eigenfunctions are, in principle, different. Comparison of these two approaches resembles the two numerical ways how to solve differential equations: finite differences employing direct approximations of derivatives in the equations, and finite elements based on variational (weak) formulations on finite-dimensional functional spaces. Our method is a modification of Hanyk *et al.* (2002) where similar ordinary differential equations are solved for post-glacial rebound, further elaborated for elastic tides of planetary bodies by Zábánová *et al.* (2009) and extended to the free-oscillations problem by Zábánová (2015). The method was already applied to analyses of the last decade biggest earthquakes in an ultralow frequency range (Zábánová *et al.* 2012a,b; Zábánová & Matyska 2014, 2016a,b).

2 GOVERNING DIFFERENTIAL EQUATIONS

The complete system of PDEs describing the response of a pre-stressed self-gravitating non-rotating elastic isotropic body to a seismic source or external gravitational force for the Lagrangian displacement $\mathbf{u}$, the Eulerian incremental gravitational potential φ and the incremental stress tensor $\boldsymbol{\tau}$ consists of the momentum equation, the Poisson equation and the constitutive relation,

$$\nabla \cdot \boldsymbol{\tau} - \rho_0 \nabla \varphi + \nabla \cdot (\rho_0 \mathbf{u}) \nabla \varphi_0 - \nabla (\rho_0 \nabla \varphi_0 \cdot \mathbf{u}) = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (1)$$

$$\nabla \cdot (\nabla \varphi + 4\pi G \rho_0 \mathbf{u}) = 0, \quad (2)$$

$$\lambda \nabla \cdot \mathbf{u} \mathbf{I} + \mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \boldsymbol{\tau}, \quad (3)$$

where ρ_0 is the reference density, φ_0 is the hydrostatic gravitational potential, G is the Newton gravitational constant and λ and μ are the elastic Lamé parameters, the latter also referred to as the shear modulus. The superscript T denotes the transposition. This Lagrangian–Eulerian system of equations can be found in, for example, Martinec (1984) or Dahlen & Tromp (1998).

Here we consider spherically symmetric models. In order to employ the spherical symmetry, we use a formalism of spherical harmonic functions for individual quantities. Let $(\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\phi)$ be the unit spherical vector basis with the radius r , the colatitude $\vartheta \in (0, \pi)$ and the longitude $\phi \in (0, 2\pi)$. On the unit sphere a quadratically integrable scalar function $f(r, \vartheta, \phi)$ can be decomposed by means of the complete orthonormal basis of scalar spherical harmonic functions $Y_{lm}(\vartheta, \phi)$,

$$f(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm}(r) Y_{lm}(\vartheta, \phi), \quad (4)$$

where l is the angular degree, m the azimuthal order and $f_{lm}(r)$ are coefficients of the expansion. We introduce the vector spherical harmonic functions $\mathcal{S}_{lm}^{(-1)} \equiv Y_{lm} \mathbf{e}_r$, $\mathcal{S}_{lm}^{(1)} \equiv \nabla_\Omega Y_{lm}$ and $\mathcal{S}_{lm}^{(0)} \equiv \mathbf{e}_r \times \nabla_\Omega Y_{lm}$. Functions $\mathcal{S}_{lm}^{(0)}$ create the toroidal basis and $\mathcal{S}_{lm}^{(-1)}$, $\mathcal{S}_{lm}^{(1)}$ form the spheroidal basis. The displacement vector $\mathbf{u}$ and traction T_r can be written in the form

$$\mathbf{u}(r) = \sum_{lm} \left[U_{lm}(r) \mathcal{S}_{lm}^{(-1)} + V_{lm}(r) \mathcal{S}_{lm}^{(1)} + W_{lm}(r) \mathcal{S}_{lm}^{(0)} \right], \quad (5)$$

$$\mathbf{e}_r \cdot \boldsymbol{\tau} \equiv T_r = \sum_{lm} \left[T_{rr,lm}(r) \mathcal{S}_{lm}^{(-1)} + T_{r\vartheta,lm}(r) \mathcal{S}_{lm}^{(1)} + T_{r\phi,lm}(r) \mathcal{S}_{lm}^{(0)} \right]. \quad (6)$$

We transform (1) into the Fourier spectral domain. In other words, we formally replace $\frac{\partial}{\partial t}$ by $i\omega$ and $\frac{\partial^2}{\partial t^2}$ by $-\omega^2$, where ω is the angular frequency of an individual free oscillation, to obtain $-\rho_0 \omega^2 \mathbf{u}$ on the right-hand side of (1). The azimuthal order m does not play a role in the case of a spherical symmetric model due to the $2l+1$ degeneracy. In what follows we denote $\zeta = l(l+1)$.

2.1 Toroidal oscillations

The radial component of the toroidal free oscillation displacements is equal to zero; moreover they create neither volume variations, nor changes of the gravitational field, that is,

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \cdot \mathbf{e}_r = 0 \quad \text{and} \quad \varphi = 0. \quad (7)$$

The toroidal displacement vector formulated by means of the vector spherical harmonic functions is expressed in terms of the functions $W \equiv W_{lm}(r)$ in (5) and eq. (1) assumes the form of an ordinary differential equation of the second order (*cf.* Martinec 1984, eq. 4.37; Dahlen & Tromp 1998, eq. 8.45):

$$\mu W'' + \left(\frac{2\mu}{r} + \mu' \right) W' - \left(\frac{\zeta \mu}{r} + \mu' \right) \frac{W}{r} = -\rho_0 \omega^2 W. \quad (8)$$

The inner boundary conditions, requiring the continuity of the displacement and the incremental traction, attain the form,

$$\begin{aligned} [W]_-^+ &= 0, \\ [T_{r\phi}]_-^+ &\equiv \left[\mu \left(W' - \frac{W}{r} \right) \right]_-^+ = 0, \end{aligned} \quad (9)$$

for the case of solid boundaries. Toroidal oscillations do not exist in liquid layers, hence the same equation holds on liquid boundaries and at the free surface,

$$T_{r\phi} \equiv \mu \left(W' - \frac{W}{r} \right) = 0. \quad (10)$$

If the model does not contain a liquid layer, toroidal displacements exist in the entire volume and conditions at the singular point $r = 0$ must be added. We show in Appendix A that the toroidal displacements vanish at the centre of the model for all modes with $l \geq 1$. We adopted the normalization of the free-oscillation eigenfunctions from Dahlen & Tromp (1998) where the kinetic energy is equal to one.

2.2 Spheroidal oscillations

The spheroidal part of the displacement vector satisfies the equation,

$$(\nabla \times \mathbf{u}) \cdot \mathbf{e}_r = 0. \quad (11)$$

Spheroidal oscillations cause volume variations; as a consequence the incremental gravitational potential φ is non-zero. Eqs (1)–(3) yield a set of three ordinary differential equations of the second order for the unknowns $U \equiv U_{lm}(r)$, $V \equiv V_{lm}(r)$ and $F \equiv F_{lm}(r)$ (cf. Martinec 1984, eqs 4.54–4.56; Dahlen & Tromp 1998, eqs 8.43–8.44, 8.53),

$$\beta U'' + \left(\frac{2\beta}{r} + \beta' \right) U' + \left(\frac{4\rho_0 g_0}{r} - 4\pi G \rho_0^2 - \frac{2\beta + \mu\zeta}{r^2} + \frac{2\lambda'}{r} \right) U - (\lambda + \mu) \frac{\zeta V'}{r} + \left(\frac{\lambda + 3\mu}{r} - \rho_0 g_0 - \lambda' \right) \frac{\zeta V}{r} - \rho_0 F' = -\rho_0 \omega^2 U, \quad (12)$$

$$\mu V'' + \left(\frac{2\mu}{r} + \mu' \right) V' - \left(\frac{\beta}{r} + \mu' \right) \frac{\zeta}{r} V + \frac{\lambda + \mu}{r} U' + \left(\frac{2\beta}{r} - \rho_0 g_0 + \mu' \right) \frac{U}{r} - \frac{\rho_0}{r} F = -\rho_0 \omega^2 V, \quad (13)$$

$$F'' + \frac{2}{r} F' - \frac{\zeta}{r^2} F + 4\pi G \rho_0 \left[U' + \left(\frac{2}{r} + \frac{\rho_0'}{\rho_0} \right) U - \frac{\zeta}{r} V \right] = 0, \quad (14)$$

where $\beta = \lambda + 2\mu$. The conditions on the inner boundaries and free surface for the displacement, incremental traction and incremental potential take the form

$$\begin{aligned} \text{solid boundary} \quad [U]_-^+ &= 0, \\ [V]_-^+ &= 0, \\ [T_{rr}]_-^+ &\equiv \left[\beta U' + \frac{\lambda}{r} (2U - \zeta V) \right]_-^+ = 0, \\ [T_{r\phi}]_-^+ &\equiv \left[\mu \left(V' - \frac{V}{r} + \frac{U}{r} \right) \right]_-^+ = 0, \\ [F]_-^+ &= 0, \\ [F' + 4\pi G \rho_0 U]_-^+ &= 0, \end{aligned} \quad (15)$$

$$\begin{aligned} \text{liquid boundary} \quad [U]_-^+ &= 0, \\ [T_{rr}]_-^+ &\equiv \left[\beta U' + \frac{\lambda}{r} (2U - \zeta V) \right]_-^+ = 0, \\ T_{r\phi} &\equiv \mu \left(V' - \frac{V}{r} + \frac{U}{r} \right) = 0, \\ [F]_-^+ &= 0, \\ [F' + 4\pi G \rho_0 U]_-^+ &= 0, \end{aligned} \quad (16)$$

$$\begin{aligned}
\text{free surface} \quad T_{rr} &\equiv \beta U' + \frac{\lambda}{r} (2U - \zeta V) = 0, \\
T_{r\vartheta} &\equiv \mu \left(V' - \frac{V}{r} + \frac{U}{r} \right) = 0, \\
Q &\equiv F' + \frac{l+1}{r} F + 4\pi G \rho_0 U = 0.
\end{aligned} \tag{17}$$

Tangential components of the traction vanish inside the liquid material, where $\mu = 0$, and thus the condition $T_{r\vartheta} = 0$ at the liquid side of the boundary is satisfied automatically. The horizontal component of the displacement V is not prescribed, and a discontinuity of V is allowed. Eq. (13) in a liquid layer simplifies to

$$\frac{\lambda}{r} U' + \left(\frac{2\lambda}{r^2} - \frac{\rho_0 g_0}{r} \right) U - \frac{\zeta \lambda}{r^2} V - \frac{\rho_0}{r} F = -\rho_0 \omega^2 V, \tag{18}$$

reducing thus the number of the second order differential equations.

The conditions at the centre of the model are (see Appendix A),

$$U(0) = V(0) = 0, \quad F(0) = 0 \quad \text{for } l > 1, \tag{19}$$

and

$$U'(0) = V'(0) = 0, \quad F(0) = 0 \quad \text{for } l = 1. \tag{20}$$

We again adopted the normalization from Dahlen & Tromp (1998) where the kinetic energy is equal to one.

2.3 Radial oscillations

If the angular degree of the spheroidal modes equals zero, $l = 0$, the motion proceeds only in the radial direction and the horizontal displacement vanishes. $F \equiv F_{00}$ satisfies (14), that arrives at a special form

$$\begin{aligned}
F'' + \frac{2}{r} F' + 4\pi G \rho_0 \left[U' + \left(\frac{2}{r} + \frac{\rho'_0}{\rho_0} \right) U \right] &= \frac{1}{r^2} (r^2 F')' + \frac{4\pi G}{r^2} (r^2 \rho_0 U)' = 0 \\
\Rightarrow F' + 4\pi G \rho_0 U &= 0,
\end{aligned} \tag{21}$$

because the integration constant is zero (Martinec 1984). Substituting (21) into (12) we obtain the ordinary differential equation of the second order for the unknown $U \equiv U_{00}$,

$$\beta U'' + \left(\frac{2\beta}{r} + \beta' \right) U' + \left(4\rho_0 g_0 - \frac{2\beta}{r} + 2\lambda' \right) \frac{U}{r} = -\rho_0 \omega^2 U. \tag{22}$$

Since the horizontal components of the displacement vector are zero, both the solid and liquid boundaries behave in the same way. The normal components of the displacement and traction have the form,

$$\begin{aligned}
\text{all internal boundaries} \quad [U]_{\pm}^+ &= 0, \\
[T_{rr}]_{\pm}^+ &\equiv \left[\beta U' + \frac{2\lambda}{r} U \right]_{\pm}^+ = 0,
\end{aligned} \tag{23}$$

$$\text{free surface} \quad T_{rr} \equiv \beta U' + \frac{2\lambda}{r} U = 0, \tag{24}$$

$$\text{the centre (see Appendix A)} \quad U(0) = 0. \tag{25}$$

In contrast to spheroidal oscillations with $l > 0$, computation of displacement is not coupled to computation of the incremental gravitational potential.

3 MATRIX REPRESENTATION

Our goal is to derive a matrix eigenvalue problem for the free oscillations. First, we arrange the unknown functions $U \equiv U_{lm}(r)$, $V \equiv V_{lm}(r)$, $F \equiv F_{lm}(r)$ and $W \equiv W_{lm}(r)$ into the vector

$$\mathbf{y}(r) = (U, V, F, W)^T. \tag{26}$$

Then eqs (8) and (12)–(14) can be rewritten into the matrix form,

$$\mathbf{A}(r) \cdot \mathbf{y}'' + \mathbf{B}(r) \cdot \mathbf{y}' + \mathbf{C}(r) \cdot \mathbf{y} = -\omega^2 \mathbf{D}(r) \cdot \mathbf{y}, \tag{27}$$

where the coefficient matrices are

$$\mathbf{A}(r) = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad (28)$$

$$\mathbf{B}(r) = \begin{pmatrix} \frac{2\beta}{r} + \beta' & -\frac{(\lambda+\mu)\zeta}{r} & -\rho_0 & 0 \\ \frac{\lambda+\mu}{r} & \frac{2\mu}{r} + \mu' & 0 & 0 \\ 4\pi G\rho_0 & 0 & \frac{2}{r} & 0 \\ 0 & 0 & 0 & \frac{2\mu}{r} + \mu' \end{pmatrix}, \quad (29)$$

$$\mathbf{C}(r) = \begin{pmatrix} \frac{4\rho_0 g_0}{r} - 4\pi G\rho_0^2 - \frac{2\beta+\mu\zeta}{r^2} + \frac{2\lambda'}{r} & \left(\frac{\lambda+3\mu}{r} - \rho_0 g_0 - \lambda'\right)\frac{\xi V}{r} & 0 & 0 \\ \frac{2\beta}{r^2} - \frac{\rho_0 g_0}{r} + \frac{\mu'}{r} & -\left(\frac{\beta}{r} + \mu'\right)\frac{\xi}{r} & -\frac{\rho_0}{r} & 0 \\ 4\pi G\left(\frac{2\rho_0}{r} + \rho'\right) & -4\pi G\rho_0\frac{\xi}{r} & -\frac{\xi}{r^2} & 0 \\ 0 & 0 & 0 & -\left(\frac{\xi\mu}{r} + \mu'\right)\frac{1}{r} \end{pmatrix}, \quad (30)$$

$$\mathbf{D}(r) = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_0 \end{pmatrix}. \quad (31)$$

In order to discretize (26) and (27) with a pseudospectral accuracy, we follow the high-order pseudospectral approach described by, for example, Fornberg (1996). The discretization grid of a layer comes out from the local extrema of the Chebyshev polynomials

$$T_{M-1}(x) = \cos[(M-1)\arccos x] \quad (32)$$

on the interval $\langle -1, 1 \rangle$,

$$x_i = \cos\left(\frac{\pi(i-1)}{M-1}\right), \quad i = 1, \dots, M. \quad (33)$$

The Chebyshev grid $r_{\min} = r_1 < \dots < r_M = r_{\max}$ thus consists of the points

$$r_{M-i+1} = r_{\min} + \frac{r_{\max} - r_{\min}}{2} \left[\cos\left(\frac{\pi(i-1)}{M-1}\right) + 1 \right], \quad i = 1, \dots, M. \quad (34)$$

The values of $\mathbf{y}$ and its derivatives at r_i can be expressed using the weighted sum of values $\mathbf{y}$ on the whole grid,

$$\mathbf{y}(r_i) = \sum_{j=1}^M \alpha_{ij} \mathbf{y}(r_j), \quad (35)$$

$$\mathbf{y}'(r_i) = \sum_{j=1}^M \beta_{ij} \mathbf{y}(r_j), \quad (36)$$

$$\mathbf{y}''(r_i) = \sum_{j=1}^M \gamma_{ij} \mathbf{y}(r_j). \quad (37)$$

The elements of weight matrices α_{ij} , β_{ij} and γ_{ij} can be evaluated by means of the algorithm developed by Fornberg (1996). Note that here, where both r_i and r_j are the grid points, the α_{ij} elements are simply $\alpha_{ii} = 1$ and $\alpha_{ij} = 0$ for $i \neq j$. With the weight matrices the equation (27) can be expressed in the form,

$$\sum_{j=1}^M [\mathbf{A}(r_i)\gamma_{ij} + \mathbf{B}(r_i)\beta_{ij} + \mathbf{C}(r_i)\alpha_{ij}] \cdot \mathbf{y}(r_j) = -\omega^2 \sum_{j=1}^M \mathbf{D}(r_i)\alpha_{ij} \cdot \mathbf{y}(r_j), \quad i = 2, \dots, M-1. \quad (38)$$

It is necessary to explain the meaning of the product, for example, $\mathbf{A}\gamma_{ij}$ in (38). The size of the matrix $\mathbf{A}$ is 4×4 , whereas γ_{ij} are elements of a matrix of the size $M \times M$. By multiplying each element of the matrix $\mathbf{A}$ by γ_{ij} we obtain a $(4M-8) \times 4M$ matrix. Eq. (38) is the discrete representation of the problem in the internal grid points. Further conditions have to be satisfied at the boundary points, r_1 and r_M , as follows.

As is summarized in Appendix A, the conditions at the centre of the model, $r_{\min} = 0$, satisfy

$$\begin{aligned} l > 1: & (U, V, F, W)^T(0) = 0, \\ l = 1: & (U', V', F, W)^T(0) = 0, \end{aligned} \quad (39)$$

and in the matrix form

$$\mathbf{F}^\beta \cdot \mathbf{y}' + \mathbf{F}^\alpha \cdot \mathbf{y} = 0, \quad (40)$$

where

$$l > 1: \mathbf{F}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{F}^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (41)$$

$$l = 1: \mathbf{F}^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{F}^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (42)$$

Using the weight matrices the boundary conditions (39) are expressed in the form

$$\sum_{j=1}^M [\mathbf{F}^\beta \beta_{0j} + \mathbf{F}^\alpha \alpha_{0j}] \cdot \mathbf{y}(r_j) = 0. \quad (43)$$

The free-surface boundary conditions (10) and (17) have the matrix form

$$(T_{rr}, T_{r\vartheta}, Q, T_{r\varphi})^T = \mathbf{A} \cdot \mathbf{y}' + \mathbf{E} \cdot \mathbf{y} = 0, \quad (44)$$

where the coefficient matrix $\mathbf{E}$ is

$$\mathbf{E}(r) = \begin{pmatrix} \frac{2\lambda}{r} & -\frac{\zeta\lambda}{r} & 0 & 0 \\ \frac{\mu}{r} & -\frac{\mu}{r} & 0 & 0 \\ 4\pi G\rho_0 & 0 & \frac{l+1}{r} & 0 \\ 0 & 0 & 0 & -\frac{\mu}{r} \end{pmatrix}, \quad r = r_M, \quad (45)$$

and with the weight matrices it can be rewritten as

$$\sum_{j=1}^M [\mathbf{A}(r_M)\beta_{Mj} + \mathbf{E}(r_M)\alpha_{Mj}] \cdot \mathbf{y}(r_j) = 0. \quad (46)$$

If we consider a one-layer model, the whole problem is thus described by eqs (43), (38) and (46) in the form of a generalized eigenvalue problem,

$$\mathbf{P} \cdot \mathbf{Y} = -\omega^2 \mathbf{R} \cdot \mathbf{Y}, \quad (47)$$

where the vector $\mathbf{Y} = (U_1 \dots U_M, V_1 \dots V_M, F_1 \dots F_M, W_1 \dots W_M)^T$ has the size $4M$ and the matrices $\mathbf{P}$ and $\mathbf{R}$ are $4M \times 4M$,

$$\mathbf{P} = \begin{pmatrix} \mathbf{F}^\beta \beta_{0j} + \mathbf{F}^\alpha \alpha_{0j} \\ \mathbf{A}\gamma_{ij} + \mathbf{B}\beta_{ij} + \mathbf{C}\alpha_{ij} \\ \mathbf{A}\beta_{Mj} + \mathbf{E}\alpha_{Mj} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 \\ \mathbf{D}\alpha_{ij} \\ 0 \end{pmatrix}, \quad (48)$$

where $i = 2, \dots, M-1$ and $j = 1, \dots, M$.

To extend the formalism to K layers, each layer with a separate Chebyshev grid, we start with the models composed of solid layers only. We mark vector $\mathbf{Y}$ as $\mathbf{Y}^{(k)}$ in the k th layer, $k = 1, \dots, K$. On the solid boundary the continuity conditions are described by (9) and (15). Using the expansions (35) and (36) we get

$$\sum_{j=1}^{M^{(k)}} \mathbf{I}\alpha_{M^{(k)}j}^{(k)} \mathbf{Y}^{(k)}(r_j^{(k)}) - \sum_{j=1}^{M^{(k+1)}} \mathbf{I}\alpha_{1j}^{(k+1)} \mathbf{Y}^{(k+1)}(r_j^{(k+1)}) = 0 \quad (49)$$

and, for brevity with layer superscripts denoting only the unknown vectors,

$$\sum_{j=1}^M [\mathbf{A}\beta_{Mj} + \hat{\mathbf{E}}\alpha_{Mj}] \mathbf{Y}^{(k)}(r_j) - \sum_{j=1}^M [\mathbf{A}\beta_{1j} + \hat{\mathbf{E}}\alpha_{1j}] \mathbf{Y}^{(k+1)}(r_j) = 0, \quad (50)$$

where $\mathbf{I}$ is the 4×4 identity matrix. The components of the coefficient matrix $\hat{\mathbf{E}}$,

$$\hat{\mathbf{E}}(r) = \begin{pmatrix} \frac{2\lambda}{r} & -\frac{\xi\lambda}{r} & 0 & 0 \\ \frac{\mu}{r} & -\frac{\mu}{r} & 0 & 0 \\ 4\pi G\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mu}{r} \end{pmatrix}, \quad (51)$$

come out from the boundary conditions (15) and (9). In fact, because of continuity of F in (15), we can use $\mathbf{E}(r)$ instead of $\hat{\mathbf{E}}(r)$.

We introduce the vector $\tilde{Y}$ composed from the unknowns $\mathbf{Y}^{(k)}$ in all layers,

$$\tilde{Y} = (\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(k)}, \dots, \mathbf{Y}^{(K)})^T. \quad (52)$$

The total number of grid points, and the size of $\tilde{Y}$, is $\tilde{M} = \sum_{k=1}^K M^{(k)}$. The matrix equation (47) generalized for K layers takes the form

$$\tilde{\mathbf{P}} \cdot \tilde{Y} = -\omega^2 \tilde{\mathbf{R}} \cdot \tilde{Y}. \quad (53)$$

The matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{R}}$ have the size $4\tilde{M} \times 4\tilde{M}$ and are given explicitly in the next sections. As the matrix $\tilde{\mathbf{P}}$ is regular and can be inverted, we arrive at the form of the standard eigenvalue problem,

$$(\tilde{\mathbf{P}}^{-1} \cdot \tilde{\mathbf{R}}) \cdot \tilde{Y} = -\frac{1}{\omega^2} \tilde{Y}. \quad (54)$$

The equations for toroidal and spheroidal motions are independent and we decouple the both parts in what follows.

3.1 Toroidal oscillations

The unknown subvector $\mathbf{y}$ for toroidal modes contains only one toroidal component of the displacement, $\mathbf{y} = (W)^T$. Also, the submatrices $\mathbf{A}_T, \dots, \mathbf{F}_T$ have one element each, corresponding to the fourth row and the fourth column of the matrices $\mathbf{A}, \dots, \mathbf{F}$. If the model contains liquid layers, where the toroidal oscillations do not exist, $r_{\min}^{(1)}$ is the radius of the first liquid boundary from the surface towards the centre and the free-surface condition (46) applies also on this boundary. If there is no liquid layer, a condition at the centre following from (43) is required instead. The $\tilde{M} \times \tilde{M}$ matrix $\tilde{\mathbf{P}}_T$ for K layers and a model with liquid layers takes the form,

$$\tilde{\mathbf{P}}_T = \begin{pmatrix} \boxed{\mathbf{A}_T \beta_{1j} + \mathbf{E}_T \alpha_{1j}} & & & & & & \\ \boxed{\mathbf{A}_T \gamma_{ij} + \mathbf{B}_T \beta_{ij} + \mathbf{C}_T \alpha_{ij}} & & & & & & \\ \boxed{\mathbf{A}_T \beta_{Mj} + \mathbf{E}_T \alpha_{Mj}} & \boxed{-\mathbf{A}_T \beta_{1j} - \mathbf{E}_T \alpha_{1j}} & & & & & \\ \boxed{\mathbf{I} \alpha_{Mj}} & \boxed{-\mathbf{I} \alpha_{1j}} & \ddots & & & & \\ & & \ddots & & & & \\ & & & \boxed{\mathbf{A}_T \gamma_{ij} + \mathbf{B}_T \beta_{ij} + \mathbf{C}_T \alpha_{ij}} & & & \\ & & & \boxed{\mathbf{A}_T \beta_{Mj} + \mathbf{E}_T \alpha_{Mj}} & & & \end{pmatrix},$$

where blocks with zero elements are left blank. The omitted layer superscript k increases along the diagonal. The matrix has a nearly block-diagonal structure and, moreover, can be inverted. Similarly, the $\tilde{M} \times \tilde{M}$ matrix $\tilde{\mathbf{R}}_T$ is

$$\tilde{\mathbf{R}}_T = \begin{pmatrix} \boxed{0}_{1 \times M} & & & & & & \\ \boxed{\mathbf{D}_T \alpha_{ij}}_{(M-2) \times M} & & & & & & \\ \boxed{0}_{1 \times M} & \boxed{0}_{1 \times M} & & & & & \\ \boxed{0}_{1 \times M} & \boxed{0}_{1 \times M} & \ddots & & & & \\ & & \ddots & & & & \\ & & & \boxed{\mathbf{D}_T \alpha_{ij}}_{(M-2) \times M} & & & \\ & & & \boxed{0}_{1 \times M} & & & \end{pmatrix}.$$

3.2 Spheroidal oscillations

The subvector $\mathbf{y}$ contains three unknown components, $\mathbf{y} = (U, V, F)^T$. The 3×3 submatrices $\mathbf{A}_S, \dots, \mathbf{F}_S$ are the upper left blocks of the matrices $\mathbf{A}, \dots, \mathbf{F}$. Spheroidal oscillations do exist in liquid layers and the point of origin, $r_{\min}^{(1)} = 0$, is always the centre of the model, where conditions (43) have to be satisfied. The free-surface conditions (46) are required on the surface. The matrices $\tilde{\mathbf{P}}_S$ and $\tilde{\mathbf{R}}_S$ are created from (43), (38), (46), (49) and (50),

$$\tilde{\mathbf{P}}_S = \begin{pmatrix} \boxed{\mathbf{F}_S^\beta \beta_{0j} + \mathbf{F}_S^\alpha \alpha_{0j}} \\ \boxed{\mathbf{A}_S \gamma_{ij} + \mathbf{B}_S \beta_{ij} + \mathbf{C}_S \alpha_{ij}} \\ \boxed{\mathbf{A}_S \beta_{Mj} + \mathbf{E}_S \alpha_{Mj}} & \boxed{-\mathbf{A}_S \beta_{1j} - \mathbf{E}_S \alpha_{1j}} \\ \boxed{\mathbf{I} \alpha_{Mj}} & \boxed{-\mathbf{I} \alpha_{1j}} & \ddots \\ & & \ddots & \boxed{\mathbf{A}_S \gamma_{ij} + \mathbf{B}_S \beta_{ij} + \mathbf{C}_S \alpha_{ij}} \\ & & & \boxed{\mathbf{A}_S \beta_{Mj} + \mathbf{E}_S \alpha_{Mj}} \end{pmatrix},$$

$$\tilde{\mathbf{R}}_S = \begin{pmatrix} \boxed{0}_{3 \times 3M} \\ \boxed{\mathbf{D}_S \alpha_{ij}}_{(3M-6) \times 3M} \\ \boxed{0}_{3 \times 3M} & \boxed{0}_{3 \times 3M} \\ \boxed{0}_{3 \times 3M} & \boxed{0}_{3 \times 3M} & \ddots \\ & & \ddots & \boxed{\mathbf{D}_S \alpha_{ij}}_{(3M-6) \times 3M} \\ & & & \boxed{0}_{3 \times 3M} \end{pmatrix}.$$

The size $3\tilde{M} \times 3\tilde{M}$ of the matrices $\tilde{\mathbf{P}}_S$ and $\tilde{\mathbf{R}}_S$ can be reduced by separation of the equations with zero right-hand sides in (53), that is, those corresponding to the Poisson equation and the boundary conditions. Thus, (53) can be arranged into the form

$$\begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_3 & \mathbf{P}_4 \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}. \quad (55)$$

$\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{R}_1$ correspond to eqs (12) and (13), whereas $\mathbf{P}_3$ and $\mathbf{P}_4$ match eq. (14) and boundary conditions (15)–(17). Vector $\mathbf{Y}_1$ is chosen to collect the displacement components in the internal points of the layers and vector $\mathbf{Y}_2$ consists of the components of the displacement in the boundary points and the incremental gravitational potential. Since

$$\mathbf{Y}_2 = -\mathbf{P}_4^{-1} \mathbf{P}_3 \mathbf{Y}_1, \quad (56)$$

the final eigenvalue problem with the matrix of the size $(2\tilde{M}-4K) \times (2\tilde{M}-4K)$ has the form

$$(\mathbf{P}_1 - \mathbf{P}_2 \mathbf{P}_4^{-1} \mathbf{P}_3)^{-1} \mathbf{R}_1 \mathbf{Y}_1 = -\frac{1}{\omega^2} \mathbf{Y}_1. \quad (57)$$

In other words, the system of linear equations that corresponds to the Poisson equation and boundary conditions is inverted separately. In this way we reduce the number of unknowns and, therefore, the size of matrices entering into numerical spectral analyses.

A remark on the calculations in fluid layers must be made. In a liquid layer eq. (18) is valid; in principle, keeping $\mu = 0$ in the matrices for liquid layers does not represent a numerical problem. There is, however, one subtle effect. If we use this equation also in a boundary point of the liquid layer in a role of a ‘boundary condition’, the number of equations corresponds to the number of unknowns on a grid, but an additional ‘discretization’ eigensolution is added and must be subsequently excluded from the spectrum. This is the reason why we use zero radial derivative of V at the liquid boundary points instead. We numerically tested that potential errors associated with this choice of the boundary condition are under numerical accuracy of calculations. Thus we get back the right number of equations at the liquid boundary points and found V from the liquid side of the boundary. The liquid layers can thus be incorporated into the calculations in a simple and straightforward way.

3.3 Radial oscillations

The subvector $\mathbf{y}$ contains only the radial component of the displacement, $\mathbf{y} = (U)$. The coefficient submatrices have one element each easily obtained from (22)–(25), that is,

$$\mathbf{A}_R \rightarrow \beta, \quad (58)$$

$$\mathbf{B}_R \rightarrow \frac{2\beta}{r} + \beta', \quad (59)$$

$$\mathbf{C}_R \rightarrow \left(4\rho_0 g_0 - \frac{2\beta}{r} + 2\lambda' \right) \frac{1}{r}, \quad (60)$$

$$\mathbf{D}_R \rightarrow \rho_0, \quad (61)$$

$$\mathbf{E}_R \rightarrow \frac{2\lambda}{r}, \quad (62)$$

$$\mathbf{F}_R^\alpha \rightarrow 1, \quad \mathbf{F}_R^\beta \rightarrow 0. \quad (63)$$

The point of origin is the centre of the model, $r_{\min}^{(1)} = 0$, where $U(0) = 0$, and the free-surface condition (24) is required on the surface. The $\tilde{M} \times \tilde{M}$ matrices $\tilde{\mathbf{P}}_R$ and $\tilde{\mathbf{R}}_R$ take the forms

$$\tilde{\mathbf{P}}_R = \begin{pmatrix} \boxed{\mathbf{F}_R^\alpha \alpha_{0j}} \\ \boxed{\mathbf{A}_R \gamma_{ij} + \mathbf{B}_R \beta_{ij} + \mathbf{C}_R \alpha_{ij}} \\ \boxed{\mathbf{A}_R \beta_{Mj} + \mathbf{E}_R \alpha_{Mj}} & \boxed{-\mathbf{A}_R \beta_{1j} - \mathbf{E}_R \alpha_{1j}} \\ \boxed{\mathbf{I} \alpha_{Mj}} & \boxed{-\mathbf{I} \alpha_{1j}} & \ddots \\ & & \ddots & \boxed{\mathbf{A}_R \gamma_{ij} + \mathbf{B}_R \beta_{ij} + \mathbf{C}_R \alpha_{ij}} \\ & & & \boxed{\mathbf{A}_R \beta_{Mj} + \mathbf{E}_R \alpha_{Mj}} \end{pmatrix},$$

$$\tilde{\mathbf{R}}_R = \begin{pmatrix} \boxed{0}_{1 \times M} \\ \boxed{\mathbf{D}_R \alpha_{ij}}_{(M-2) \times M} \\ \boxed{0}_{1 \times M} & \boxed{0}_{1 \times M} \\ \boxed{0}_{1 \times M} & \boxed{0}_{1 \times M} & \ddots \\ & & \ddots & \boxed{\mathbf{D}_R \alpha_{ij}}_{(M-2) \times M} \\ & & & \boxed{0}_{1 \times M} \end{pmatrix}.$$

4 QUALITY FACTORS AND DISPERSION

Anelastic dissipation of the compressional and shear energy causes the attenuation of the free oscillations. The modal amplitude decreases with time proportionally to $\exp(-\frac{\omega t}{2Q})$, where Q is the quality factor of a mode. When Q is large, the mode is attenuated slowly. In the Earth, the quality factors of the modes vary from a few tens to 5500. However, only the first few radial modes achieve values higher than 1000, whereas Q of other modes are lower.

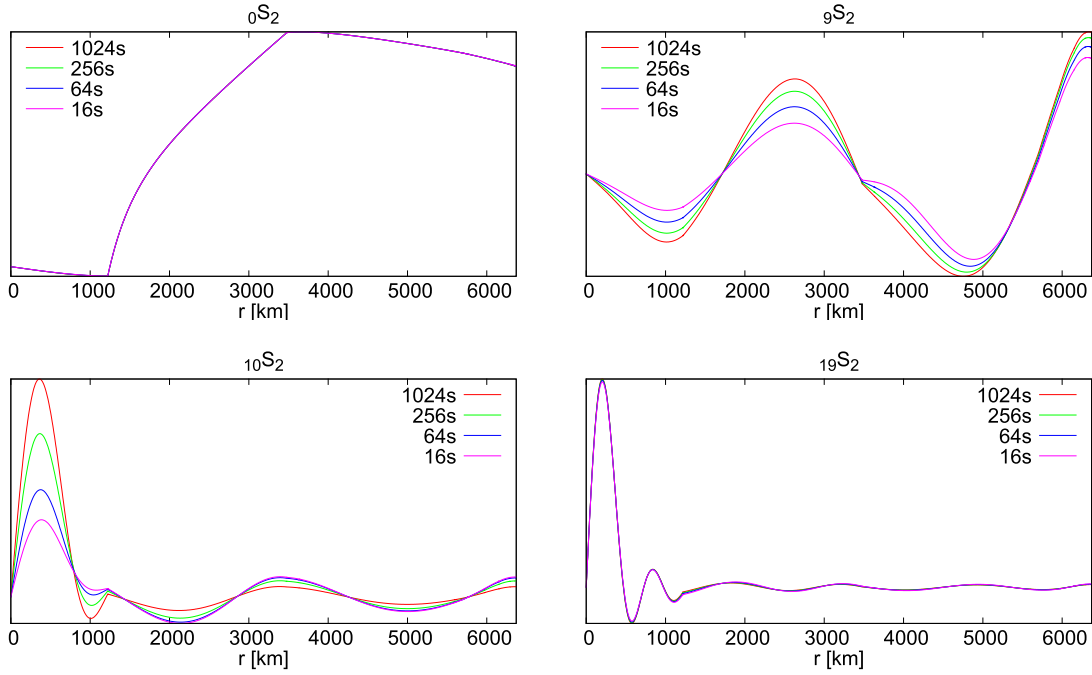


Figure 1. The radial eigenfunctions U of selected overtones ${}_nS_2$ for several reference periods of the PREM model.

The value of Q can be determined by the equation (Dahlen & Tromp 1998, eq. 9.54)

$$Q^{-1} = 2\omega^{-1} \int_0^R (\kappa K_\kappa Q_\kappa^{-1} + \mu K_\mu Q_\mu^{-1}) dr, \quad (64)$$

where $\kappa = \lambda + \frac{2}{3}\mu$ is the incompressibility (bulk) modulus, $Q_\kappa(r)$ and $Q_\mu(r)$ are the bulk and shear quality factors of the model and the Fréchet kernels K_κ and K_μ depend on the modal frequencies and eigenfunctions. In our calculations we use Q_κ and Q_μ from the PREM (Dziewonski & Anderson 1981) but there is a series of other different models (e.g. Sailor & Dziewonski 1978; Widmer *et al.* 1991; Durek & Ekström 1996; Lawrence & Wyssession 2005).

The seismic velocities $v_p = \sqrt{(\kappa + \frac{4}{3}\mu)/\rho}$ and $v_s = \sqrt{\mu/\rho}$ are treated as frequency dependent in realistic Earth models. For example, the PREM velocities are given at the reference frequency of $\omega_{\text{ref}}/(2\pi) = 1$ Hz and have to be modified according to dispersion relations by Kanamori & Anderson (1977),

$$v_s(\omega) = v_s(\omega_{\text{ref}}) \left(1 + \frac{\ln \frac{\omega}{\omega_{\text{ref}}}}{\pi Q_\mu} \right), \quad (65)$$

$$v_p(\omega) = v_p(\omega_{\text{ref}}) \left[1 + \frac{\ln \frac{\omega}{\omega_{\text{ref}}}}{\pi} \left(\frac{1-E}{Q_\kappa} + \frac{E}{Q_\mu} \right) \right], \quad (66)$$

where $E = \frac{4}{3}(v_s/v_p)^2$. Frequency-dependent values of $\lambda(\omega)$ and $\mu(\omega)$ stem from these relations easily.

Note that some spheroidal eigenfunctions are very sensitive to changes of seismic velocity. Fig. 1 shows the frequency dependence of eigenfunctions U for several overtones of the angular degree $l=2$. The eigenfrequencies and eigenfunctions were computed for the PREM model for four selected periods (1024 s, 256 s, 64 s, 16 s). Whereas the eigenfunctions of the fundamental mode ${}_0S_2$ are almost independent of the choice of the PREM period, the eigenfunctions of the modes ${}_9S_2$ and ${}_{10}S_2$ are quite sensitive. Since Q depends on the eigenfunctions, its values vary between 510.4 and 509.3 for the mode ${}_0S_2$, 523.4 and 313.6 for the mode ${}_9S_2$, 112.6 and 535.8 for the mode ${}_{10}S_2$. The former values correspond to the model period of 1024 s and the latter to that of 16 s. The dependence obtained from calculations for nine PREM periods is shown in Fig. 2.

Dispersion of the model due to the frequency dependence of velocity thus represents a further problem (Tromp & Dahlen 1990). We decided to avoid numerical obstacles with solving non-linear eigenvalue problems. Instead we deal with linear problems, where the material parameters in the equations are fixed for several chosen fiducial frequencies ω_0 of the model. In this aspect, classical integration in shooting methods is more straightforward as, in principle, the frequency-dependent parameters of the model can be re-evaluated in each shot. In our method the question arises how to correct the eigenfrequencies different from the fiducial one and corresponding eigenfunctions.

Dahlen & Tromp (1998, eq. 9.55) derived the explicit formula for the first-order perturbation $\delta\omega_d$ of the eigenfrequency ω ,

$$\delta\omega_d = \omega \frac{\ln \frac{\omega}{\omega_0}}{\pi Q}. \quad (67)$$

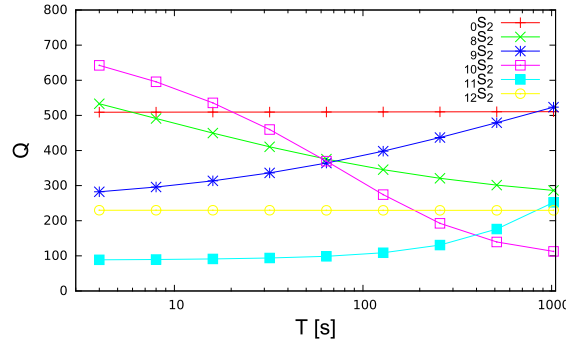


Figure 2. Dependence of the quality factors on PREM reference periods.

If T_1 and T_2 are the modal periods corresponding to the fiducial periods T_1^f and T_2^f of the model, $T_1^f < T_2^f$, we start from the logarithmic dependence of eigenfrequency perturbation and similarly use the ad hoc approximative relation for the corrected modal period,

$$T = T_1 + \frac{\log((T_1 + T_2)/2) - \log T_1^f}{\log T_2^f - \log T_1^f} (T_2 - T_1). \quad (68)$$

The analogous relations can be written for the eigenfunctions U , V and F ,

$$U = U_1 + \frac{\log T - \log T_1^f}{\log T_2^f - \log T_1^f} (U_2 - U_1), \quad \dots \quad (69)$$

and for the quality factors Q ,

$$Q = Q_1 + \frac{\log T - \log T_1^f}{\log T_2^f - \log T_1^f} (Q_2 - Q_1). \quad (70)$$

If a modal quality factor Q is low (below 100), the correction $\delta\omega_d$ can be so high that the sequence of the overtones is disarranged. For this reason we compute the distance integral $\int_0^R [(U_1 - U_2)^2 + (V_1 - V_2)^2] \rho r^2 dr$ to evaluate the correspondence of the overtones. If Q obtained from (70) agrees well with the value yielded by (64), we consider these modes to be corresponding. There are exceptional cases (less than 0.1 per cent of the total number of the modes up to 250 mHz) when we are not able to find the corresponding mode rising from neighbouring fiducial periods. In such cases we do not interpolate and take the modal periods and eigenfunctions from the solution on the closer fiducial frequency.

5 NUMERICAL TESTS

The matrices of the standard eigenvalue problems (54), resp. (57), are real, non-sparse and non-symmetric. We computed their eigenvalues and eigenvectors with the routine `devcrg` of the IMSL library (<https://www.roguewave.com/products-services/imsl-numerical-libraries>), based on the routine `dgeev` of the LAPACK library (<http://www.netlib.org/lapack>) and built on top of the Intel Math Kernel Library (Intel MKL, <https://software.intel.com/en-us/intel-mkl>). This solver returned more robust results than the LAPACK routine `dggev` applied to the generalized eigenvalue problem (53); actually, poor behaviour of `dggev` motivated us to reformulate (53) into (54). The sparse structure of matrices in (53) for layered models is lost with this transition but LAPACK eigenvalue solvers do not provide specific support for non-symmetric band matrices anyway.

The tests and benchmarks we describe below were performed with the IMSL `devcrg` solver. Evaluation of efficiency and accuracy of solvers from other libraries, for example, FEAST (<http://www.ecs.umass.edu/~polizzi/feast>), is postponed for future work. Note that the series of matrix eigenvalue problems can be trivially solved in parallel, one for each angular degree l .

We tested the accuracy of the computed eigenfrequencies ω by comparing them with numerically obtained Rayleigh quotients R_q (e.g. Dahlen & Tromp 1998, pp. 280 and 290) that should amount to ω^2 ,

$$\text{toroidal modes: } R_q \equiv \frac{E_\mu}{E_k} = \omega^2, \quad \text{spheroidal modes: } R_q \equiv \frac{E_\kappa + E_\mu + E_g}{E_k} = \omega^2. \quad (71)$$

For each mode, $\omega^2 E_k$, E_κ , E_μ and E_g denote the kinetic, compressional, shear and gravitational energy that can be found from the eigenfunctions U , V , W and F . Numerical agreement of R_q and ω^2 , specifically the relative error $|(\sqrt{R_q} - \omega)/\omega|$, can thus be used as a measure of internal accuracy of our method for a given mode. Relative errors for the modes with eigenfrequencies up to 500 mHz computed for the PREM model with the fiducial period of 16 sec are presented in Fig. 3. For the toroidal modes, we have obtained low relative errors of 1 per cent. Relative errors for almost all computed spheroidal modes do not exceed 2.5 per cent. However, three modal branches emerged with relative errors up to 5 per cent (branches 1 and 2 in Fig. 3) or 10 per cent (branch 3 in Fig. 3). Eigenfunctions of some of these modes are revealed in Fig. 4. Branches 1 and 2 consist of the CMB and ICB Stoneley modes, respectively. Modes in branch 3 are discretization modes since the eigenfunctions V oscillate between adjacent grid points within the liquid core and are thus dependent on discretization.

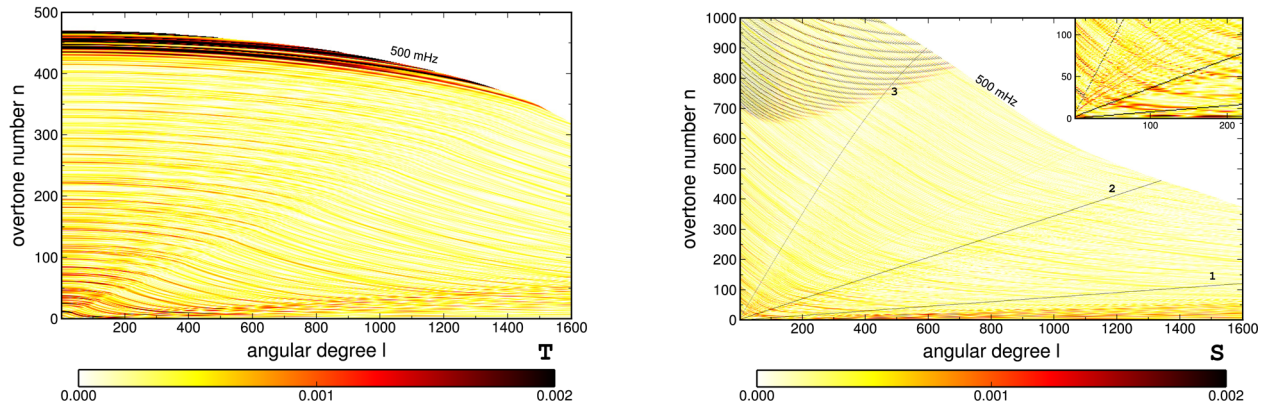


Figure 3. Relative errors of frequencies of the toroidal (left panel) and spheroidal (right panel) modes up to 500 mHz estimated by means of the Rayleigh quotients calculated for the PREM model.

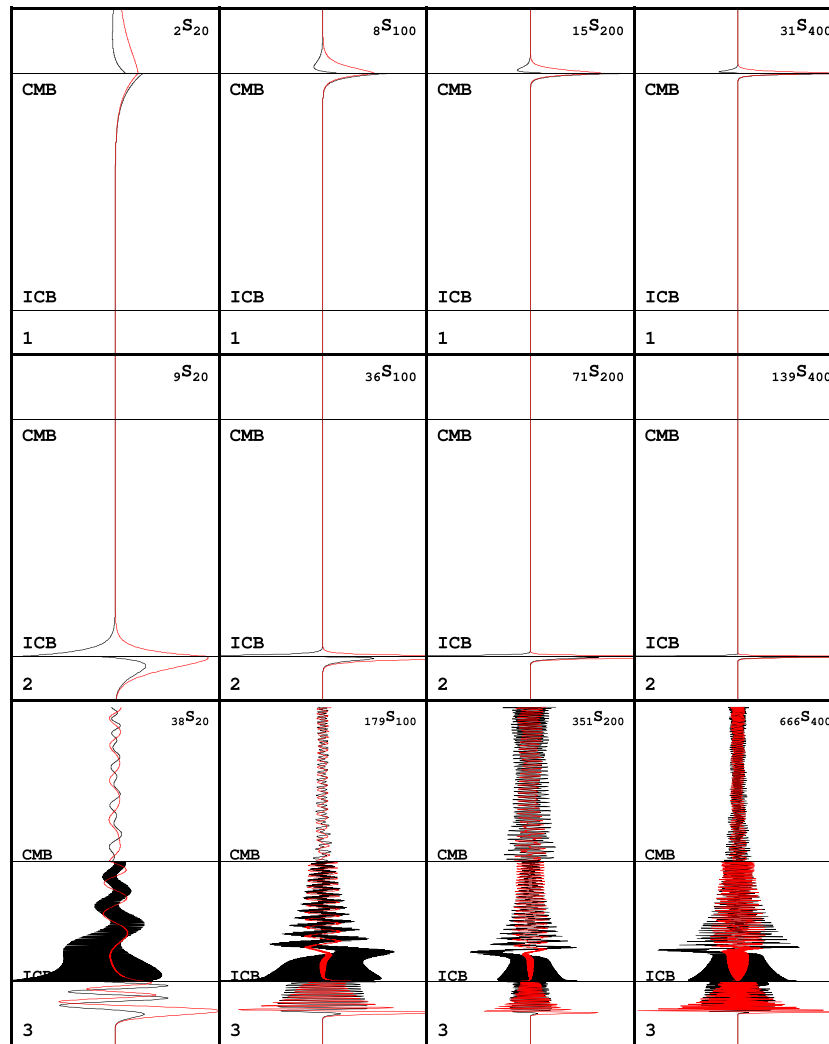


Figure 4. Eigenfunctions U (red) and V (black) of selected PREM modes from the branches 1, 2 and 3 in Fig. 3.

We prescribed a total of about 2000 Chebyshev non-equidistant grid points for spheroidal modes and about 1000 grid points for toroidal modes to guarantee more than 6 grid points per a wavelength in each layer and to compute frequencies of the overtones up to 500 mHz with a good accuracy. Note that modes of branch 3 do not satisfy this criterion and are, therefore, automatically excluded during post-processing. The number of grid points in each of the PREM layers is at least 80. With this grid resolution, the matrix eigenvalue problem for spheroidal modes of one angular degree is solved in about 10 minutes using one core of the CPU i7/3.2 GHz, and the CPU time depends cubically on

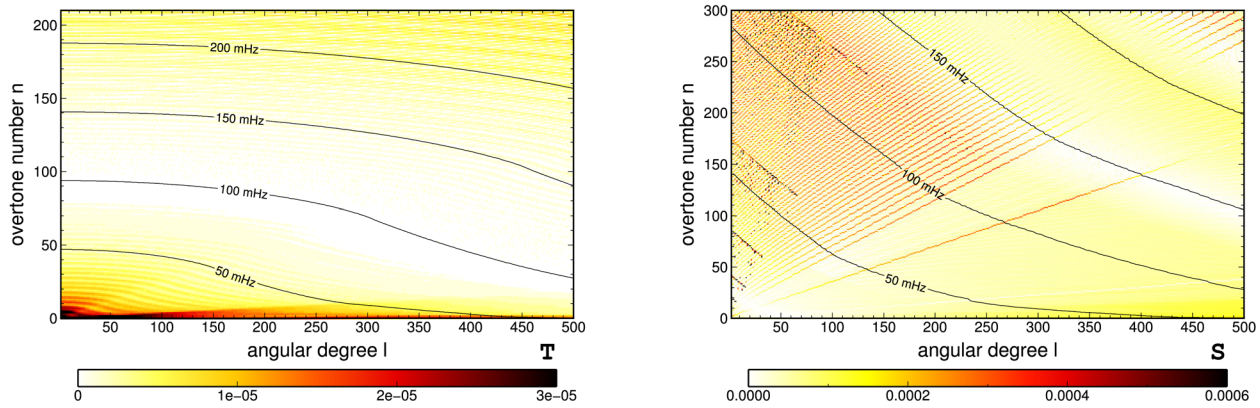


Figure 5. Relative errors between the frequencies of the toroidal (left panel) and spheroidal (right panel) modes obtained by Mineos and our matrix eigenvalue software for the angular degrees up to 500.

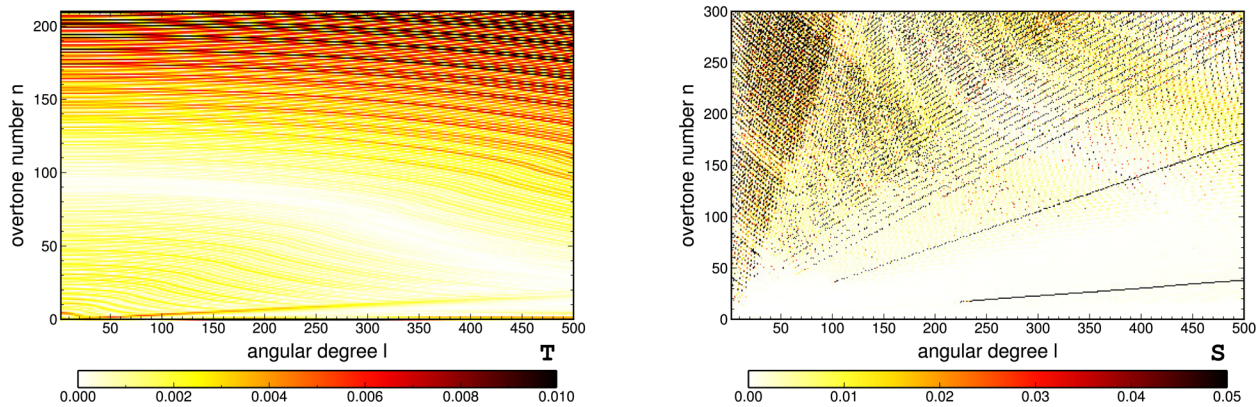


Figure 6. Relative errors between the quality factors of the toroidal (left panel) and spheroidal (right panel) modes obtained by Mineos and our matrix eigenvalue software for the angular degrees up to 500.

the number of grid points. Approximately one fourth of the numerical spectrum is occupied by the undertones excited in the fluid core. No undertones are plotted in Fig. 3 as they do not have practical applications in seismology.

Next, we benchmarked our spectra with the results computed by the Mineos software package (<http://geodynamics.org/cig/software/mineos>). We again selected the resolution for our software to about 2000 grid points for spheroidal modes and about 1000 grid points for toroidal modes. The PREM parameters were evaluated with the fiducial frequency of 100 mHz for toroidal modes and with several fiducial frequencies with subsequent interpolation for spheroidal modes. The Mineos package was configured to neglect the gravitational terms for frequencies above 40 mHz. We show the relative errors of the eigenfrequencies (Fig. 5) and quality factors (Fig. 6) for the toroidal and spheroidal modes. All modes with angular degrees up to 500 and overtones with $n \leq 300$ for spheroidal modes and $n \leq 210$ for toroidal modes are included. This span guaranties that all modes up to 50 mHz are included into the benchmark; most of the frequencies are higher but all below 250 mHz. CPU times by the Mineos package are substantially shorter than those of our software with the selected resolution; however, we often faced freezing of Mineos runs at higher frequencies.

We achieved excellent agreement for all toroidal modes since the choice of the fiducial frequency plays a minor role here. We obtained relative errors of the eigenfrequencies less than 10^{-5} for most of the modes and less than 10^{-7} for the modes with frequencies near 100 mHz. Discrepancy of the toroidal quality factors can increase up to 1 per cent in the worst scenario. As to the spheroidal modes, the relative errors of the frequencies are less than 10^{-4} for most of the modes and the worst relative error is about 10^{-3} . The coincidence of quality factors is better than 95 per cent for all spheroidal modes that converged in Mineos.

The last test yields comparison of the response of an elastic non-rotating spherical Earth model to a seismic event calculated again using Mineos and our software. The acceleration response of this model to a step-function moment-tensor source $\mathbf{M}$ situated at $\mathbf{x}_s$ is given at the location of receiver $\mathbf{x}$ by

$$\mathbf{a}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} {}_n\mathbf{A}_l(\mathbf{x}) \cos({}_n\omega_l t) \exp\left(-\frac{{}_n\omega_l t}{2{}_nQ_l}\right). \quad (72)$$

The formulae for evaluating the amplitudes ${}_n\mathbf{A}_l(\mathbf{x})$ can be found in Dahlen & Tromp (1998), section 10.3.

Fig. 7 shows the three-component records of the synthetic acceleration following the 2010 Maule earthquake calculated on a non-rotating sphere for the station Pecný (PE: 49.92N, 14.79E), Czech Republic. The Global CMT Project Moment Tensor Solution (GCMT) with $M_0 = 1.84 \times 10^{22}$ Nm ($M_{rr} = 1.04 \times 10^{22}$ Nm, $M_{\theta\theta} = -0.03 \times 10^{22}$ Nm, $M_{\phi\phi} = -1.01 \times 10^{22}$ Nm, $M_{r\theta} = 0.227 \times 10^{22}$ Nm,

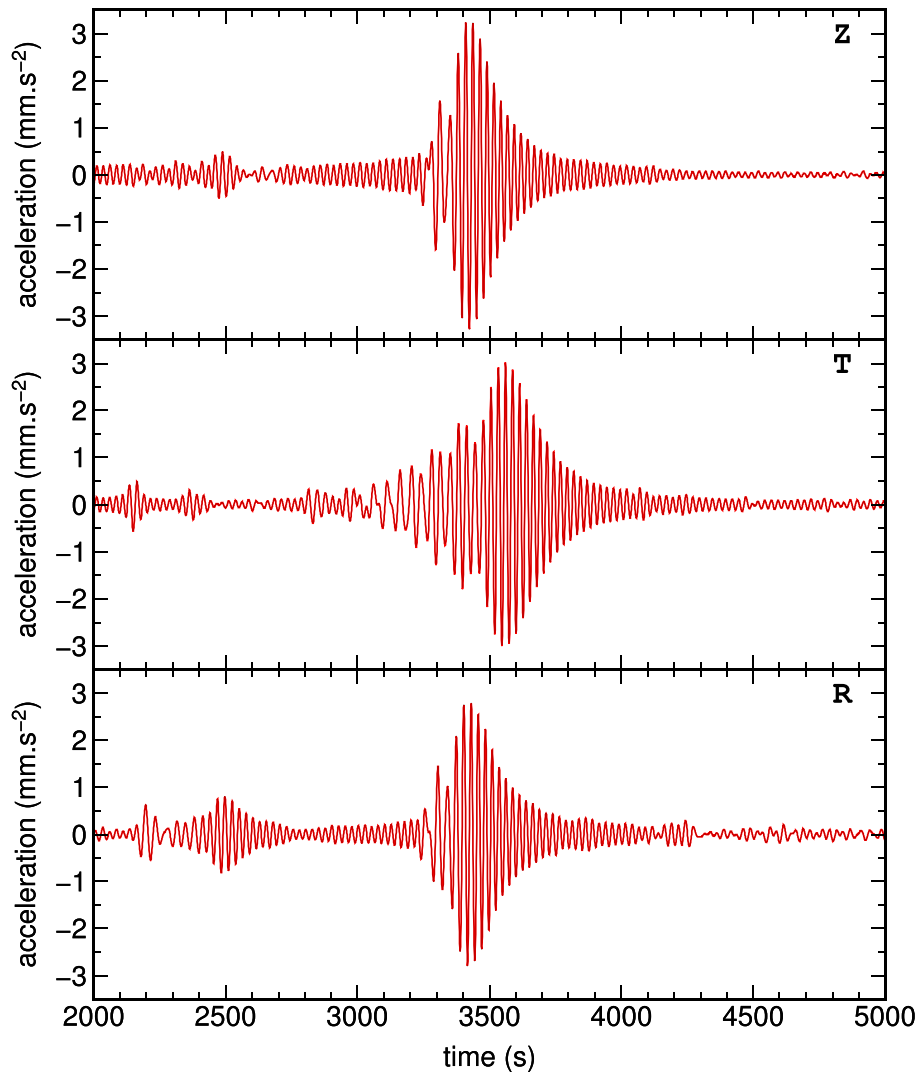


Figure 7. Three components (vertical Z, transverse T, radial R) of synthetic accelerograms for the spheroidal and toroidal modes up to 40 mHz generated by the Maule 2010 earthquake at the station PE (49.92N, 14.79E). The GCMT source solution was chosen to represent the earthquake. Our accelerograms are plotted in red, those calculated using Mineos in black. Note that the (Z, R)-components (dominated by the minor-arc R1 Rayleigh waves) are the sums of the spheroidal modes, whereas the T-component (dominated by the minor-arc G1 Love waves) is formed of toroidal modes. The zero of the time axis corresponds to the origin time of the earthquake.

$M_{r\phi} = -1.51 \times 10^{22}$ Nm and $M_{\theta\phi} = -0.12 \times 10^{22}$ Nm) is used as the seismic source; the centroid location is 35.95S and 73.15W and depth of 24.1 km. The synthetic signals up to 40 mHz are computed by our code and Mineos. The vertical component Z as well as the radial component R are generated by the spheroidal modes, whereas the transverse component T is formed of toroidal modes. There is an excellent agreement between the accelerograms obtained by these two substantially different numerical approaches. Note that the difference is less than 1 per cent of the amplitude also for several following packages of surface waves.

6 CONCLUDING REMARKS

We have demonstrated that the systems of ordinary differential equations describing free oscillations of spherically symmetric elastic prestressed self-gravitating models can be directly discretized by means of finite-difference schemes with pseudospectral accuracy. The original problem can thus be transformed to a series of matrix eigenvalue problems that can be solved by standard numerical libraries for matrix spectral analysis. The size of matrices is reduced as we discretize the systems of ordinary differential equations of the second order and not those of the first order. Moreover, the subsystem of equations describing the Poisson equation and boundary conditions for the spheroidal modes is inverted separately.

The eigenfunctions and eigenfrequencies of the fundamental mode and corresponding overtones are simultaneously obtained and a high accuracy up to a discretization limit is achieved. We have also shown that liquid layers can be incorporated into the models in a direct simple way, where special attention must be paid only to the boundary condition for horizontal displacement at the boundary from the liquid side.

If frequency dependence of a model is taken into account, it results, in principle, into non-linear eigenproblems. We have avoided these non-linear problems by solving several linear problems on fixed frequencies with subsequent interpolation of the results.

The internal test based on evaluation of the Rayleigh quotients up to 500 mHz demonstrated that the accuracy of the method is satisfactory. Eigenfrequencies and eigenfunctions calculated from our discretized systems also agree with those obtained from the classical shooting methods, as represented by the Mineos software package based on integration techniques for governing systems of ordinary differential equations. Note that the advantage of the classical shooting methods is that such integration is performed for a fixed frequency and changes of a model due to frequency dependence follow changes of integration frequencies. On the other hand, the upper limit of the presented method is limited only by a resolution of the Chebyshev grids and, consequently, by sizes of the obtained matrices that enter into the numerical eigenanalyses. The potential frequency range of our numerical method can thus increase with increasing hardware and software power.

ACKNOWLEDGEMENTS

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APPENDIX A: CONDITIONS AT THE CENTRE OF THE MODEL

The centre of the Earth model, $r = 0$, represents the singular point and the question arises how to include it into numerical calculations. One of the classical approaches is to employ analytical solutions for a homogeneous sphere (expressed by means of the spherical Bessel functions) in

the vicinity of the centre and start numerical integration at a small radial distance (e.g. Takeuchi & Saito 1972; Dahlen & Tromp 1998). Here we analyse solutions of the corresponding second-order ordinary differential equations by means of power series of the second order and study the behaviour of coefficients of these series at the neighbourhood of the centre. A similar approach was already employed by Crossley (1975) and Martinec (1984) but they analysed the systems of first-order differential ordinary equations for displacements, tractions and gravity field perturbations. The analysis presented here is designed for our method based on discretization of the second-order differential equations.

A1 Toroidal oscillations

The eigenfunctions W_n decomposed into the power series get the form for $r \rightarrow 0$,

$$W_l = a_0 + a_1 r + a_2 r^2 + O(r^3). \quad (\text{A1})$$

By substitution of this expression into (8), we obtain the relation

$$(6 - \zeta) a_2 + \frac{2 - \zeta}{r} a_1 - \frac{\zeta a_0}{r^2} = -\frac{\rho_0 \omega^2}{\mu} a_0, \quad (\text{A2})$$

which has three different branches dependent on the angular degree l . We abbreviate $\zeta = l(l + 1)$.

(1) $l = 1$ and $\zeta = 2$

Expansion coefficients a_0 and a_2 must be equal to zero to satisfy (A2). Thus, we have conditions for the zeroth and second derivatives of toroidal displacement,

$$W_1(0) = W_1''(0) = 0. \quad (\text{A3})$$

(2) $l = 2$ and $\zeta = 6$

We obtain conditions for the zeroth and first derivatives of W_2 , since $a_0 = a_1 = 0$,

$$W_2(0) = W_2'(0) = 0. \quad (\text{A4})$$

(3) $l > 2$ and $\zeta > 6$

Since $a_0 = a_1 = a_2 = 0$, all derivatives vanish in the centre,

$$W_l(0) = W_l'(0) = W_l''(0) = 0. \quad (\text{A5})$$

A2 Radial oscillations

The expansions of the displacement and incremental gravitational potential in the case of $l = 0$ have the form for $r \rightarrow 0$,

$$U_0 = a_0 + a_1 r + a_2 r^2 + O(r^3), \quad (\text{A6})$$

$$F_0 = c_0 + c_1 r + O(r^2). \quad (\text{A7})$$

By substitution of these expressions into (21) and (22) and using $g_0(r) = \frac{4}{3}\pi G\rho_0 r$, we obtain the system of equations

$$4\beta a_2 + \left(\frac{16}{3}\pi G\rho_0^2 - \frac{2\beta}{r^2}\right) a_0 = -\rho_0 \omega^2 a_0, \quad (\text{A8})$$

$$c_1 + 4\pi G\rho_0 a_0 = 0, \quad (\text{A9})$$

which yields $a_0 = 0$, $c_1 = 0$, $a_2 = 0$, and results in the conditions

$$U_0(0) = U_0''(0) = F_0'(0) = 0. \quad (\text{A10})$$

A3 Spheroidal oscillations

In the case of spheroidal oscillations, $l > 0$, we have three eigenfunctions represented by the power-series expansions,

$$U_l = a_0 + a_1 r + a_2 r^2 + O(r^3), \quad (\text{A11})$$

$$V_l = b_0 + b_1 r + b_2 r^2 + O(r^3), \quad (\text{A12})$$

$$F_l = c_0 + c_1 r + c_2 r^2 + O(r^3). \quad (\text{A13})$$

By substitution into the (12)–(14) and using $g_0(r) = \frac{4}{3}\pi G\rho_0 r$, we obtain the system of equations for $r \rightarrow 0$,

$$(4\beta - \zeta\mu)a_2 - \frac{\zeta\mu}{r}a_1 + \left(\frac{4}{3}\pi G\rho_0^2 - \frac{2\beta + \zeta\mu}{r^2}\right)a_0 + \zeta(\mu + \lambda)b_2 + \frac{2\zeta\mu}{r}b_1 + \zeta\left(\frac{3\mu + \lambda}{r^2} - \frac{4}{3}\pi G\rho_0^2\right)b_0 - \rho_0 c_1 = -\rho_0\omega^2 a_0, \quad (\text{A14})$$

$$(4\lambda + 6\mu)a_2 + \frac{3\lambda + 5\mu}{r}a_1 + \left(\frac{2\beta}{r^2} - \frac{4}{3}\pi G\rho_0^2\right)a_0 + (6\mu - \zeta\beta)b_2 + \frac{2\mu - \zeta\beta}{r}b_1 - \frac{\beta\zeta}{r^2}b_0 - \rho_0 c_1 - \frac{\rho_0}{r}c_0 = -\rho_0\omega^2 b_0, \quad (\text{A15})$$

$$4\pi G\rho_0\left(3a_1 + \frac{2}{r}a_0 - \zeta b_1 - \frac{\zeta}{r}b_0\right) + (6 - \zeta)c_2 + \frac{2 - \zeta}{r}c_1 - \frac{\zeta}{r^2}c_0 = 0. \quad (\text{A16})$$

Similarly to the case of the toroidal oscillations, we have three different branches.

(1) $l = 1$ and $\zeta = 2$

From (A14) we obtain

$$(2\beta + 2\mu)a_0 = 2(\lambda + 3\mu)b_0$$

$$\Rightarrow a_0 = b_0$$

$$-2\mu a_1 + 4\mu b_1 = 0$$

$$\Rightarrow a_1 = 2b_1$$

$$(4\lambda + 6\mu)a_2 - \frac{4}{3}\pi G\rho_0^2 a_0 - (2\lambda - 2\mu)b_2 - \rho_0 c_1 = -\rho_0\omega^2 a_0$$

From (A14) and (A15) it follows that

$$2\beta a_0 - 2\beta b_0 = 0$$

$$\Rightarrow a_0 = b_0$$

$$2(3\lambda + 5\mu)b_1 + (2\mu - 2\beta)b_1 - \rho_0 c_0 = 0$$

$$\Rightarrow b_1 = \frac{\rho_0}{4\beta}c_0$$

$$(4\lambda + 6\mu)a_2 - \frac{4}{3}\pi G\rho_0^2 a_0 - (2\lambda - 2\mu)b_2 - \rho_0 c_1 = -\rho_0\omega^2 a_0.$$

From (A16) we get

$$2c_0 = 0$$

$$\Rightarrow c_0 = 0$$

$$4\pi G\rho_0(2a_0 - 2b_0) = 0$$

$$\Rightarrow a_0 = b_0$$

$$4\pi G\rho_0(3a_1 - 2b_1) + 4c_2 = 0$$

$$\Rightarrow a_1 = -\frac{c_2}{2\pi G\rho_0}$$

It follows that $c_0 = 0 \Rightarrow b_1 = 0 \Rightarrow a_1 = 0 \Rightarrow c_2 = 0$, and

$$U_1'(0) = V_1'(0) = F_1(0) = F_1''(0) = 0, \quad U_1(0) = V_1(0). \quad (\text{A17})$$

For $l = 1$, (A14) and (A15) are identical. Thus, we have only one equation constraining the remaining coefficients,

$$\rho_0\left(\omega^2 - \frac{4}{3}\pi G\rho_0\right)a_0 + (4\lambda + 6\mu)a_2 - (2\lambda - 2\mu)b_2 - \rho_0 c_1 = 0.$$

(2) $l = 2$ and $\zeta = 6$

From (A14) we obtain

$$(-2\lambda - 10\mu)a_0 + (6\lambda + 18\mu)b_0 = 0$$

$$\Rightarrow a_0 = b_0 = 0$$

$$-6\mu a_1 + 12\mu b_1 = 0$$

$$\Rightarrow a_1 = 2b_1$$

$$(4\lambda + 2\mu)a_2 + 6(\mu + \lambda)b_2 - \rho_0 c_1 = 0.$$

From (A14) and (A15) it follows that

$$a_0 = b_0 = 0$$

$$\begin{aligned}
(3\lambda + 5\mu)2b_1 + (2\mu - 6\beta)b_1 - \rho_0 c_0 &= 0 \\
&\Rightarrow c_0 = 0 \\
(4\lambda + 6\mu)a_2 - 6(\mu + \lambda)b_2 - \rho_0 c_1 &= 0.
\end{aligned}$$

From (A16) we get

$$\begin{aligned}
c_0 &= 0 \\
c_1 &= 0 \\
a_1 &= 2b_1.
\end{aligned}$$

We get $a_0 = b_0 = c_0 = c_1 = 0$, b_1 and c_2 can be arbitrary and $a_1 = 2b_1$, consequently $a_2 = b_2 = 0$, and, finally,

$$U_2(0) = U_2'(0) = V_2(0) = V_2''(0) = F_2(0) = F_2''(0) = 0, \quad U_2'(0) = 2V_2'(0). \quad (\text{A18})$$

(3) $l > 2$ and $\zeta > 6$

From (A14) we obtain

$$\begin{aligned}
-(2\beta + \mu\zeta)a_0 + (3\mu + \lambda)\zeta b_0 &= 0 \\
&\xRightarrow{b_0 = \frac{2}{\zeta}a_0} a_0 = b_0 = 0, \text{ if } \mu > 0 \\
-\mu\zeta a_1 + 2\mu\zeta b_1 &= 0 \\
&\Rightarrow a_1 = 2b_1, \text{ if } \mu > 0 \\
(4\beta - \mu\zeta)a_2 + \frac{4}{3}\pi G\rho_0^2(a_0 - \zeta b_0) - (\lambda - \mu)\zeta b_2 - \rho_0 c_1 &= -\rho_0\omega^2 a_0.
\end{aligned} \quad (\text{A19})$$

From (A14)–(A15) it follows that

$$\begin{aligned}
-\beta\zeta b_0 + 2\beta a_0 &= 0 \\
&\Rightarrow a_0 = \frac{\zeta}{2}b_0 \\
(2\mu - \beta\zeta)b_1 + (3\lambda + 5\mu)a_1 - \rho_0 c_0 &= 0 \\
&\xRightarrow{c_0=0, a_1=2b_1} a_1 = b_1 = 0 \\
(6\mu - \beta\zeta)b_2 + (4\lambda + 6\mu)a_2 - \frac{4}{3}\pi G\rho_0^2 a_0 - \rho_0 c_1 &= -\rho_0\omega^2 b_0.
\end{aligned} \quad (\text{A20})$$

From (A16) we get

$$\begin{aligned}
c_0 &= 0 \\
(2 - \zeta)c_1 + (2a_0 - \zeta b_0)4\pi G\rho_0 &= 0 \\
&\xRightarrow{a_0 = \frac{\zeta}{2}b_0} c_1 = 0, \text{ if } \mu > 0 \\
(6 - \zeta)c_2 + (3a_1 - \zeta b_1)4\pi G\rho_0 &= 0 \\
&\xRightarrow{a_1=b_1=0} (c_2 + 4\pi G\rho_0 b_1) = 0, \text{ if } \mu > 0.
\end{aligned}$$

We get $a_0 = a_1 = b_0 = b_1 = c_0 = c_1 = c_2 = 0$ and from (A19)–(A20) we obtain $a_2 = b_2 = 0$. Thus, finally,

$$U_n(0) = U_n'(0) = U_n''(0) = V_n(0) = V_n'(0) = V_n''(0) = F_n(0) = F_n'(0) = F_n''(0) = 0. \quad (\text{A21})$$