

On Helmert's 2nd condensation method

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Abstract

In the past, two different methods were proposed to consider the effect of the terrain in Helmert's 2nd condensation method. In Vaníček and Kleusberg (1987) approach the attraction of the topographical masses is evaluated at a point on the topographical surface. By analogy with the Molodensky's theory, Wang and Rapp (1990) claim that the free-air anomaly should be reduced by the terrain correction. They also state that the attraction of the topographical masses should be referred to a point on the geoid.

This paper shows that key to solve this discrepancy is hidden in the way how the downward continuation of the anomalous gravity is treated in the particular methods. In the Vaníček and Kleusberg approach (1987) the downward continuation of the anomalous gravity from the topographical surface to the geoid is completely neglected, whereas Wang and Rapp (1990) evaluate this term under implicit assumption that there is a linear relationship between free-air gravity anomaly and the elevation of topography. As Moritz (1966) showed such an assumption comes from a simplified view of the compensation of topographical masses; it has not been proved yet that this assumption is acceptable for a precise geoid determination.

In other words it means that both methods, Vaníček and Kleusberg (1987) as well as Wang and Rapp (1990), are for different reasons only approximate. We will not be able to decide which method yields more accurate results until a correct procedure of computing downward continuation of anomalous gravity will be employed. Let us emphasize that this paper does not aspire to provide such an accurate procedure.

Introduction

Stokes's formula for the gravimetric determination of the geoid requires that there be no masses outside the geoid and the gravity anomaly be referred to the geoid. One way how to satisfy these requirements is to use the Helmert 2nd condensation technique (Heiskanen and Moritz, 1967, sects. 3-7, 4-3; Vaníček and Kleusberg, 1987; Sideris and Forsberg, 1990). This condensation technique is applied as follows:

- (1) To replace the effect of topographical masses on gravity at the earth surface by the effect of the mass layer on the geoid; this difference is known as the **direct topographical effect on gravity** (ibid.);
- (2) To compute the gravity anomaly at the geoid by evaluating actual gravity at the geoid from the observed surface value through **downward continuation**;
- (3) Now Stokes's technique can be applied since the above 2 requirements are satisfied to give the so-called **co-geoid**;
- (4) To compute the **indirect topographical effect on potential**, (ibid.), to obtain the geoid.

According to Sideris and Forsberg (1990, eqn.(17)) steps (1) and (2) yield Δg^0 on the geoid computed from the formula

$$\Delta g^0 = g_P - \gamma_Q + F - A_P + A_{P_g}^c, \quad (1)$$

where g_P is the observed gravity at the surface point P , γ_Q is the normal gravity at a point Q on the reference ellipsoid, F is the free-air reduction at the point P , A_P is the attraction of the topographical masses (above the geoid) at the point P , and $A_{P_g}^c$ is the attraction of the condensed topography at the point P_g on the geoid.

The aim of the paper is to answer two questions which occur when the Helmert condensation technique is applied. The first concerns the term $A_{P_g}^c$. Heiskanen and Moritz (1967, sect. 4-3), Sideris and Forsberg (1990, eqn.(17)), and Wang and Rapp (1990, Conclusion) claim that the attraction of the condensed topographical masses must be referred to the point P_g on the geoid as denoted by the subscript. Vaníček and Kleusberg (1987, eqn.(10)) claim that the attraction of the condensed topography must be computed at the point P on the topographical surface, i.e they replace the term $A_{P_g}^c$ by a term A_P^c .

The second problem concerns the downward continuation of the gravity anomalies from the topographical surface to the geoid. In the Vaníček and Kleusberg (1987) approach this step is performed by the simple free-air reduction of $-0.3086H$ mGal/m. This term represents the downward continuation of the gravity in free-air when the gravity is represented only by a spherical harmonic of degree zero. A question arises how the gravity data are distorted by the free-air reduction, if they also contain the spherical harmonics of degrees higher than zero. In other words, up to which spherical harmonic degree is it possible to approximate the actual vertical gradient of gravity by the simple free-air reduction?

Boundary condition for the anomalous potential

In this section we derive the boundary condition for the anomalous gravitational potential in the case when the Helmert 2nd condensation technique is employed. Let us start with a decomposition of the potential of the gravitational field generated by the Earth. It may be split into two parts:

$$V = V^g + V^t, \quad (2)$$

where V^g is the potential generated by the masses below the geoid and V^t is the potential generated by the topographical masses (masses between the geoid and the topographical surface). The potential of the topographical masses may be further decomposed as

$$V^t = V^c + \delta V, \quad (3)$$

where V^c is the potential of the masses condensed on the geoid with the density $\sigma = \rho H$, H being the orthometric height of the topography above the point of interest. The term δV is obviously the residual potential given by the difference between the actual potential of the topographical masses and the potential of the condensed masses.

Inserting eqn.(3) into eqn.(2), we get

$$V = V^h + \delta V, \quad (4)$$

where

$$V^h = V^g + V^c \quad (5)$$

may be considered as a potential approximating the actual potential V . As shown in Martinec and Vaníček (1992), the effect on geoidal undulations by the topographical potential V^t is of the order of 10^3 m, whereas the residual potential δV influences geoidal heights of only about 2m. That is why, the residual potential δV may be computed from relatively much less precise expressions, e.g., by evaluating the Newton volume integral using approximate estimates of the topographical density. Such estimates cannot be used in computing the condensation potential V^c , because they would cause errors in geoidal heights 3 orders of magnitude larger. Therefore the potential V^c together with the potential V^g will be considered as unknown quantities—see eqn.(7) below. Since the potentials V^g and V^c are harmonic outside the geoid, the potential V^h is obtained by solving the Laplace equation, $\nabla^2 V^h = 0$ outside the geoid, with a boundary condition prescribed on it.

Keeping in mind eqn.(4), the gravity potential of the Earth takes the form

$$W = V^h + \delta V + \Phi, \quad (6)$$

where Φ is the centrifugal potential of the Earth. The gravity potential $V^h + \Phi$ may be written as a sum of a normal gravity potential U generated by a biaxial ellipsoid spinning with the same angular velocity as the earth and a (unknown) disturbing potential T^h ,

$$V^h + \Phi = U + T^h. \quad (7)$$

The superscript 'h' emphasizes that T^h approximates the actual disturbing potential T of the Earth, $T = W - U$. The difference between T^h and T ,

$$\delta V = T - T^h, \quad (8)$$

measured in geoidal heights reaches at most 2m (Wichiencharoen, 1982, Martinec and Vaníček, 1992). Inserting (7) into (6), the gravity potential W outside the geoid becomes

$$W = U + T^h + \delta V, \quad (9)$$

where

$$\nabla^2 T^h = 0 \quad \text{outside the geoid.} \quad (10)$$

Let us apply the operator $\text{grad}\{\cdot\}$ to eqn.(9) and take the magnitude of the resulting vector. We get

$$|\text{grad}W| = |\text{grad}U| + \frac{\text{grad}U \cdot \text{grad}(T^h + \delta V)}{|\text{grad}U|} + O\left(\frac{|\text{grad}(T^h + \delta V)|^2}{|\text{grad}U|}\right), \quad (11)$$

where ' $\cdot$ ' denotes the scalar product of vectors. The radial component of the normal gravity vector $\text{grad}U$

is 3×10^2 larger than its horizontal components. To an accuracy better than $1 \mu\text{Gal}$ (Vaníček and Martínez, 1993) eqn.(11) becomes

$$g \doteq \gamma - \frac{\partial T^h}{\partial r} - \frac{\partial \delta V}{\partial r}, \quad (12)$$

where $g = |\text{grad}W|$ is the actual gravity and $\gamma = |\text{grad}U|$ is the normal gravity. Using eqn.(3), the last term in eqn.(12) becomes

$$\frac{\partial \delta V}{\partial r} = \frac{\partial V^t}{\partial r} - \frac{\partial V^c}{\partial r} = -A + A^c, \quad (13)$$

where A is the radial component of the attraction of the topographical masses and A^c is the radial component of the attraction of the condensed topography. Now, eqn.(12) reads

$$\frac{\partial T^h}{\partial r} = -g + \gamma + A - A^c. \quad (14)$$

This equation is valid everywhere above the geoid.

Let us consider eqn.(14) at a point P on the topographical surface,

$$\left. \frac{\partial T^h}{\partial r} \right|_P = -g_P + \gamma_P + A_P - A_P^c. \quad (15)$$

Normal gravity γ_P may be expressed by means of γ and its derivatives at the point Q on the reference ellipsoid on the same geocentric radius as the point P (Heiskanen and Moritz, 1967, sect. 2-14),

$$\begin{aligned} \gamma_P &= \gamma_Q + \left. \frac{\partial \gamma}{\partial r} \right|_Q (N + H) + \dots \\ &\doteq \gamma_Q - 2 \frac{\gamma_Q}{R} N - F, \end{aligned} \quad (16)$$

where R is the mean radius of the Earth, F is the free-air reduction,

$$F = - \left. \frac{\partial \gamma}{\partial r} \right|_Q H, \quad (17)$$

and N is the geoidal height given by the Bruns formula (Heiskanen and Moritz, 1967, eqn.(2.144)),

$$N = \frac{1}{\gamma_Q} (T^h + \delta V)|_{P_g}. \quad (18)$$

Here P_g is the point on the geoid that lies on the same geocentric radius as points P and Q . Inserting eqns.(16) and (18) into eqn.(15) yields

$$\left. \frac{\partial T^h}{\partial r} \right|_P + \frac{2}{R} T_{P_g}^a = -\Delta g_P^F + A_P - A_P^c - \delta_s, \quad (19)$$

where

$$\Delta g_P^F = g_P - \gamma_Q + F \quad (20)$$

is the free-air gravity anomaly (Vaníček and Krakiwsky, 1986, eqn.(21.14)). The term

$$\delta_s = - \left. \frac{\partial \gamma}{\partial r} \right|_Q \frac{\delta V_{P_g}}{\gamma_Q} \doteq \frac{2}{R} \delta V_{P_g} \quad (21)$$

is called the secondary indirect effect on gravity (Heiskanen and Moritz, 1967, eq.(3-51)).

To obtain the radial derivatives of potential T^h on the geoid, as required in Stokes's integration, let us develop the radial derivatives of T^h at the topographical surface into the Taylor series. Since T^h is harmonic between topographical surface and geoid, we can write:

$$\left. \frac{\partial T^h}{\partial r} \right|_P = \left. \frac{\partial T^h}{\partial r} \right|_{P_g} + \left. \frac{\partial^2 T^h}{\partial r^2} \right|_{P_g} H + \dots \quad (22)$$

Taking only the first two terms of this Taylor series expansion, equation (19) takes the final form

$$\left. \frac{\partial T^h}{\partial r} \right|_{P_g} + \frac{2}{R} T_{P_g}^a = -\Delta g^H, \quad (23)$$

which is the spherical approximation of the boundary condition for the disturbing potential T^h at the point P_g on the geoid. The Helmert anomaly reads

$$\Delta g^H = \Delta g_P^F - A_P + A_P^c + g_1 + \delta_s, \quad (24)$$

where

$$g_1 = \left. \frac{\partial^2 T^h}{\partial r^2} \right|_{P_g} H. \quad (25)$$

Now, the two requirements of Stokes's integration are satisfied: the boundary condition (23) is referred to a point on the geoid, and the disturbing potential T^h is harmonic outside the geoid. Therefore, Stokes's integration may be immediately applied to eqn.(23).

Comparing eqn.(24) with (1), we can see that the attraction A_P^c of the condensed topography must be referred to the point P on the topographical surface as was claimed by Vaníček and Kleusberg (1987) and not to the point P_g on the geoid as was argued, e.g., by Sideris and Forsberg (1990), and Wang and Rapp (1990).

Helmert's anomaly contains the term g_1 which does not appear in eqn.(1). Since $\partial^2 T^h / \partial r^2$ is the vertical gradient of the anomalous gravitation $\partial T^h / \partial r$, the term $g_1 = (\partial^2 T_{P_g}^a / \partial r^2) H$ represents the harmonic downward continuation of the anomalous gravitation from P to P_g . In the Vaníček and Kleusberg (1987) approach the downward continuation of the gravity is only realized by the simple free-air reduction and the term g_1 is neglected. But if the gravity contains high-frequency components, the term g_1 may be significant and has to be considered in Helmert's condensation reduction procedure.

Moritz's terrain correction approximation

In this section we will discuss the term g_1 in detail. The quantity g_1 may be written as follows:

$$g_1 = \left. \frac{\partial^2 T^h}{\partial r^2} \right|_{P_g} H = \frac{\partial}{\partial r} \left(\frac{\partial T^h}{\partial r} + \frac{2}{R} T^h - \frac{2}{R} T^h \right) \Big|_{P_g} H = - \frac{\partial \Delta g^H}{\partial r} \Big|_{P_g} H - 2 \frac{H}{R} \frac{\partial T^h}{\partial r} \Big|_{P_g} \quad (26)$$

The vertical gradient of gravity appearing in eqn.(26) may be expressed by means of eqn.(2-217) in (Heiskanen and Moritz, 1967). We get

$$g_1 = 2 \frac{H}{R} \left(\Delta g^H - \frac{\partial T^h}{\partial r} \right) \Big|_{P_g} - \frac{R^2}{2\pi} H \int_{\Omega'} \frac{\Delta g_{P_g}^H(\Omega') - \Delta g_{P_g}^H}{\ell_0^3} d\Omega' \quad (27)$$

where ℓ_0 is the spherical distance between the dummy point in the integration and the point of interest (ibid., Fig.1-13). In planar approximation (Moritz, 1980, eqn. (45-31)), the first term on the right hand side of eqn.(27) may be neglected, so the downward continuation term g_1 is approximately equal to

$$g_1 \doteq - \frac{R^2}{2\pi} H \int_{\Omega'} \frac{\Delta g_{P_g}^H(\Omega') - \Delta g_{P_g}^H}{\ell_0^3} d\Omega' \quad (28)$$

This expression may be called the 'gradient solution' of the downward continuation in analogy with the gradient solution used in Molodensky's theory (Moritz, 1980, sect.45).

To be able to compute the term g_1 , the gravity anomalies are to be continued from the topographical surface (Δg_P^H) to the geoid ($\Delta g_{P_g}^H$). To avoid this procedure, Pellinen (1962) and after him also Moritz (1966) suggested to assume that the free-air gravity anomaly is linearly dependent on the elevation H of topography. Let us assume that this assumption is also valid for the Helmert anomaly,

$$\Delta g^H = a + 2\pi G \rho H \quad (29)$$

where a is a constant and ρ is the density of the topographical masses. Under this assumption, the downward continuation term g_1 becomes

$$g_1 = -G \rho R^2 H \int_{\Omega'} \frac{H(\Omega') - H}{\ell_0^3} d\Omega' \quad (30)$$

Now, the correction terms A_P , A_P^c and g_1 that appear in Helmert's anomaly (24) depend on topographical height H only, and may be thus added together.

Provided that $\ell_0 \gg H$, Vaníček and Kleusberg (1987, eqn.(14)) derived that

$$-A_P + A_P^c = \frac{1}{2} G \rho R^2 \int_{\Omega'} \frac{H^2(\Omega') - H^2}{\ell_0^3} d\Omega' \quad (31)$$

The sum of eqns.(30) and (31) yields

$$-A_P + A_P^c + g_1 = C \quad (32)$$

where

$$C = \frac{1}{2} G \rho R^2 \int_{\Omega'} \frac{[H(\Omega') - H]^2}{\ell_0^3} d\Omega' \quad (33)$$

is the Moritz terrain correction (Moritz, 1980, sect.48). Under the assumption that the Helmert gravity anomaly is linearly dependent on the elevation of topography, the Helmert anomaly reads

$$\Delta g^H = \Delta g_P^F + C + \delta_s \quad (34)$$

The sum of the free-air anomaly Δg_P^F plus the terrain correction C may be called a refined Faye anomaly (cf., Faye, 1883). Equation (34) means that Helmert's anomaly is equal to a refined Faye anomaly (except the small term δ_s) provided that the assumption (29) is valid.

Conclusion

This paper investigates one of the questionable points of Helmert's condensation technique concerning the evaluation of the attraction of the topographical masses. From our derivations it seems clear that this term has to be applied at a point on the topographical surface as was proposed by Vaníček and Kleusberg (1987).

Provided that the Helmert anomaly is linearly dependent on the terrain elevation, we have shown that the terrain correction term

$$\frac{1}{2} G \rho R^2 \int_{\Omega'} \frac{H^2(\Omega') - H^2}{\ell_0^3} d\Omega' \quad (35)$$

employed in the Vaníček and Kleusberg (1987) approach is replaced by the Moritz terrain correction term

$$\frac{1}{2} G \rho R^2 \int_{\Omega'} \frac{[H(\Omega') - H]^2}{\ell_0^3} d\Omega' \quad (36)$$

used in the Wang and Rapp (1990) approach. The last term represents a sum of the terrain correction term (35) and the downward continuation term g_1 , cf. eqn.(32). This means that downward continuation term g_1 is missing in the Vaníček and Kleusberg (1987) approach whereas it is treated approximately under the assumption (29) by Wang and Rapp (1990).

A questionable point of the Wang and Rapp approach (1990) remains the Pellinen assumption about

linear relationship between free-air gravity anomalies and topographical heights. Moritz (1966) showed that this assumption corresponds to a simple compensation model of the topographical masses. There are areas on the earth where the topographical masses are compensated according to this simple model. In such areas, Wang and Rapp advocated approach may yield more accurate geoidal heights than the Vaníček and Kleusberg approach. But today's knowledge about the structure of the uppermost parts of the earth points out that there are regions where the compensation mechanism of the topographical masses is rather complicated and is dependent on different physical phenomena. In such areas the assumption about linear relationship between free-air gravity anomalies and topographical heights remains questionable. The accuracy of geoidal heights over such areas coming from Wang and Rapp approach can be worse than that of Vaníček and Kleusberg approach, particularly, in cases when the mass distribution in the earth creates the gravity field which behaves opposite to the assumption (29).

A logical consequence of the facts above is that the discussion wheather the Wang and Rapp (1990) approach or the Vaníček and Kleusberg (1987) approach is better does not make any sense. Both methods are approximate for different reasons and we cannot decide at this point which of them yields more accurate results. Only by applying a correct procedure for treating the downward continuation of anomalous gravity, we will be able to solve this controversy.

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