

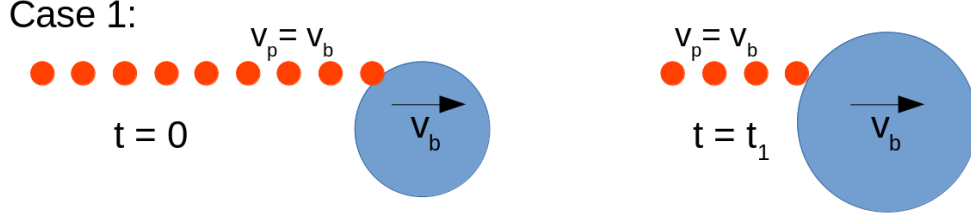
Newton's second law of motion - variable mass system

Below, it is argued that a reasonable form of the second law of motion is $\mathbf{F} = m \frac{d\mathbf{v}}{dt}$, while $\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$ is generally not valid – it is only a special form that is correct only when the velocity of the mass entering the studied system changes from $\mathbf{0}$ to $\mathbf{v}$ during the merging process.

A trivial case where

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{dm}{dt}\mathbf{v} + m \frac{d\mathbf{v}}{dt} \quad (1)$$

results in an unacceptable solution is the following (labeled Case 1):



Assume a body of mass m_b moving with velocity $\mathbf{v}_b$ in an inertial reference frame with no resistance ($\mathbf{F} = \mathbf{0}$). Such a body moves with a constant velocity $\mathbf{v}_b$. Next, assume that the body is being chased by small particles of mass m_p whose velocity, due to whatever external forcing (irrelevant here), slows down to exactly $\mathbf{v}_b$ upon reaching the body. When the body is reached, they become one with the body – the mass of the system increases during the first encounter from m_b to $m_b + m_p$ and so on, I label the total mass as $m_t(t)$ (Fig. 1). Clearly, the body should continue moving with the constant velocity $\mathbf{v}_b$, because all that is changing is the mass of what we consider to be the system of interest ($\frac{dm_t}{dt} = \frac{dN}{dt} m_p$ where $N(t)$ is the number of already attached m_p particles). However, applying Eq. (1) to this case,

$$\mathbf{0} = \frac{dm_t}{dt} \mathbf{v}_b + m_t \frac{d\mathbf{v}_b}{dt} \quad (2)$$

results in $\frac{d\mathbf{v}_b}{dt} \neq 0$, which is wrong. This shows that Eq. (1) cannot be applied in all circumstances, and it is thus not a generally valid law. In fact, Eq. (2) correctly describes a different experiment, labeled Case 2: Experiment in which the body encounters static particles m_p on its way and loses momentum in perfectly inelastic collisions with these small particles (Fig. 2). In this case, Eq. (2) correctly captures how the body slows down due to the inelastic interactions.



In summary, the general form of the second law of motion is

$$\mathbf{F} = \frac{dm}{dt} \Delta \mathbf{v} + m \frac{d\mathbf{v}}{dt} \quad (3)$$

where $\Delta \mathbf{v}$ is the velocity change of the incoming/outcoming mass to the system. In other words, $\frac{dm}{dt} \Delta \mathbf{v}$ must correctly account for in/out-flux of momentum into/from the system.

One can check the above conclusion by integrating the equation of motion assuming the change in mass is constant in time, labeled $\dot{m}$ (hence $m_t = m_0 + \dot{m}t$ with m_0 the initial mass of the body). The equation of motion then results in:

$$\frac{d\mathbf{v}_b}{dt} = -\frac{\dot{m}}{m_0 + \dot{m}t}\mathbf{v}_b \Rightarrow \mathbf{v}_b(t) = \frac{m_0 v_{b0}}{m_0 + \dot{m}t} \quad (4)$$

This solution, however, is what one would obtain from the conservation of momentum of the entire system consisting of the body and all the particles in Case 2:

$$(m_0 + \dot{m}t) \mathbf{v}_b = m_t(t) \mathbf{v}_b(t) = m_0 v_{b0}, \quad (5)$$

while in Case 1 the momentum of the body + all particles is not simply given as $m_0 v_{b0}$, i.e. as the initial momentum of the body.

Eq. (3) can be derived when $\mathbf{F} = m \frac{d\mathbf{v}}{dt}$ is applied to the individual constituents of the joined system. One can play with their own definitions of what is the (variable mass) body, which determines $\frac{dm}{dt}$, but, of course, the resulting solution is (and must be) independent of such choice.

Note that the textbook examples where Newton's second law is applied in the special (generally not correct) form (1) can be divided into two groups:

i) The law is, in fact, applied to a system with a constant mass, and division into "body" and the rest is introduced only later, which then gives equation of motion for the body. This is the classic case of a rocket losing fuel as it accelerates:

$$\mathbf{F} = \frac{d(\mathbf{p}_{\text{rocket}} + \mathbf{p}_{\text{gas}})}{dt} = m_{\text{rocket}} \frac{d\mathbf{v}_{\text{rocket}}}{dt} + \frac{dm_{\text{rocket}}}{dt} \mathbf{v}_{\text{rocket}} + \frac{dm_{\text{gas}}}{dt} (\mathbf{v}_{\text{rocket}} - \mathbf{v}_{\text{rel}}) = m_{\text{rocket}} \frac{d\mathbf{v}_{\text{rocket}}}{dt} - \frac{dm_{\text{gas}}}{dt} \mathbf{v}_{\text{rel}}$$

ii) The incoming mass does in fact have $\Delta \mathbf{v} = \mathbf{v}$ (e.g. Buquoy's problems).

In the above Cases 1 and 2, a discrete model was used. Of course, one can extend the thought experiments to the continuum mechanics framework by transforming the discrete m_p particles into a continuum with some density. Note that in the theory of mixtures, where continuum mechanics of variable mass systems are thoroughly analyzed (individual components of the mixture are variable mass systems when e.g. chemical reaction is taking place), the in/out-flux of momentum is carefully accounted for: when there is a mass transfer between two components of the mixture, which move with different velocities, it is necessarily accompanied by a corresponding linear momentum transfer, and the conservation assumption secures that the total momentum of both components does not change in this process.