

Cvičení 1 – Souřadnicový zápis, úvod

Geometrický přístup

- skalár (invariant)
- vektor (tečna ke křivce)
- forma (gradient funkce)
- tenzor (bilineární zobrazení)

vs.

Složkový zápis

„automatizace“ výpočtu
počítačová implementace

Kartézské souřadnice

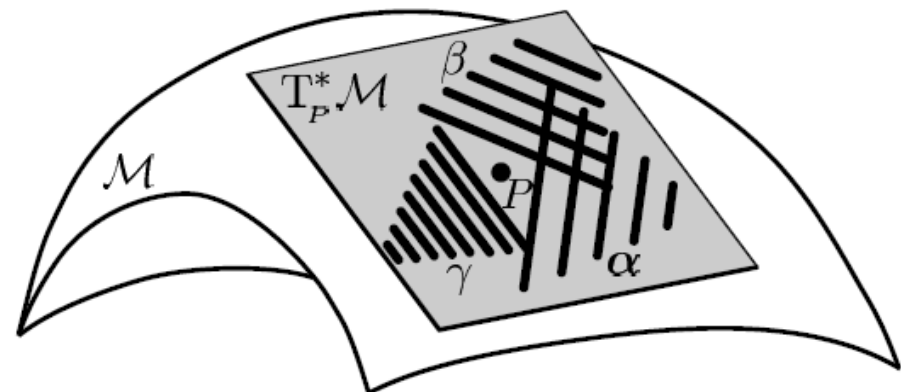
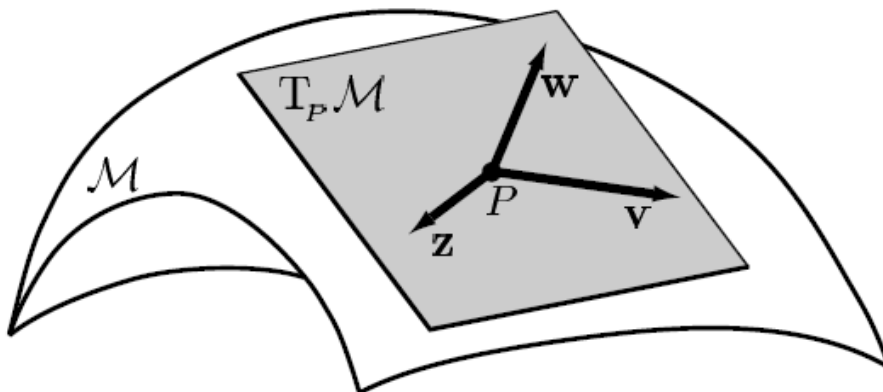
- netřeba rozlišovat vektory a formy
- ortonormální bázi lze zkonstruovat i pro křivočaré souřadnice, ale bázové vektory se potom mění v prostoru

vs.

Křivočaré souřadnice

Vazba na diferenciální geometrii (NOFY003)

- zjednodušení (indexy všechny dole)



Cvičení 1 - Vektory a tenzory v E^3

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Vektorová ortonormální báze

$\vec{e}_\alpha, \quad \alpha = 1, 2, 3$ vektorová báze

$\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$ skalární součin

$\vec{e}_\beta \times \vec{e}_\gamma = \epsilon_{\alpha\beta\gamma} \vec{e}_\alpha$ vektorový součin (pravotočivá báze)

Tenzorová ortonormální báze

$\mathbf{e}_{\alpha\beta} = \vec{e}_\alpha \otimes \vec{e}_\beta$ tenzorová báze

$$\mathbf{e}_{\alpha\beta} \cdot \vec{e}_\gamma = (\vec{e}_\alpha \otimes \vec{e}_\beta) \cdot \vec{e}_\gamma = \vec{e}_\alpha (\vec{e}_\beta \cdot \vec{e}_\gamma) = \delta_{\beta\gamma} \vec{e}_\alpha$$

$$\vec{e}_\gamma \cdot \mathbf{e}_{\alpha\beta} = \vec{e}_\gamma \cdot (\vec{e}_\alpha \otimes \vec{e}_\beta) = (\vec{e}_\gamma \cdot \vec{e}_\alpha) \vec{e}_\beta = \delta_{\alpha\gamma} \vec{e}_\beta$$

$$\vec{e}_\alpha \cdot \mathbf{e}_{\beta\gamma} \cdot \vec{e}_\delta = \vec{e}_\alpha \cdot (\vec{e}_\beta \otimes \vec{e}_\gamma) \cdot \vec{e}_\delta = (\vec{e}_\alpha \cdot \vec{e}_\beta) (\vec{e}_\gamma \cdot \vec{e}_\delta) = \delta_{\alpha\beta} \delta_{\gamma\delta}$$

$$\mathbf{e}_{\alpha\beta} \cdot \mathbf{e}_{\gamma\delta} = (\vec{e}_\alpha \otimes \vec{e}_\beta) \cdot (\vec{e}_\gamma \otimes \vec{e}_\delta) = (\vec{e}_\beta \cdot \vec{e}_\gamma) (\vec{e}_\alpha \cdot \vec{e}_\delta) = \delta_{\beta\gamma} (\vec{e}_\alpha \cdot \vec{e}_\delta)$$

$$\mathbf{e}_{\alpha\beta} : \mathbf{e}_{\gamma\delta} = (\vec{e}_\alpha \otimes \vec{e}_\beta) : (\vec{e}_\gamma \otimes \vec{e}_\delta) = (\vec{e}_\beta \cdot \vec{e}_\gamma) (\vec{e}_\alpha \cdot \vec{e}_\delta) = \delta_{\beta\gamma} \delta_{\alpha\delta}$$

Vztah Levi-Civitova symbolu ke Kroneckerovu delta

$$\epsilon_{ijk} \epsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{pmatrix}$$

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl}$$

$$\epsilon_{ijk} \epsilon_{ijk} = 6$$

Vektory a vektorové operace

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 = v_k \vec{e}_k$$

Skalární součin

$$\vec{v} \cdot \vec{u} = (v_i \vec{e}_i) \cdot (u_k \vec{e}_k) = v_i u_k (\vec{e}_i \cdot \vec{e}_k) = v_k u_k$$

Vektorový součin

$$\begin{aligned} \vec{v} \times \vec{u} &= \epsilon_{ijk} \vec{e}_i v_j u_k = (v_2 u_3 - v_3 u_2) \vec{e}_1 \\ &+ (v_3 u_1 - v_1 u_3) \vec{e}_2 + (v_1 u_2 - v_2 u_1) \vec{e}_3 \end{aligned}$$

Dyadický součin

$$\begin{aligned} \vec{v} \otimes \vec{u} &= (v_i \vec{e}_i) \otimes (u_j \vec{e}_j) = v_i u_j (\vec{e}_i \otimes \vec{e}_j) = \\ &= v_i u_j \mathbf{e}_{ij} = \begin{pmatrix} v_1 u_1 & v_1 u_2 & v_1 u_3 \\ v_2 u_1 & v_2 u_2 & v_2 u_3 \\ v_3 u_1 & v_3 u_2 & v_3 u_3 \end{pmatrix} \end{aligned}$$

... a převod na maticové násobení:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (\text{matice})_{3 \times 1} \quad \vec{v}^T = (v_1 \ v_2 \ v_3) = (\text{matice})_{1 \times 3}$$

$$\begin{aligned} \vec{v} \cdot \vec{u} &= \vec{v}^T \vec{u} = (v_1 \ v_2 \ v_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (\text{matice } v)_{1 \times 3} (\text{matice } u)_{3 \times 1} \\ &= \text{matice}_{1 \times 1} = \text{skalár} = v_1 u_1 + v_2 u_2 + v_3 u_3 \end{aligned}$$

$$\vec{v} \times \vec{u} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\begin{aligned} \vec{v} \otimes \vec{u} &= \vec{v} \vec{u}^T = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} (u_1 \ u_2 \ u_3) = (\text{matice } v)_{3 \times 1} (\text{matice } u)_{1 \times 3} \\ &= \text{matice}_{3 \times 3} = \begin{pmatrix} v_1 u_1 & v_1 u_2 & v_1 u_3 \\ v_2 u_1 & v_2 u_2 & v_2 u_3 \\ v_3 u_1 & v_3 u_2 & v_3 u_3 \end{pmatrix} \end{aligned}$$

Tenzory

$$\mathbf{T} = T_{ij} \mathbf{e}_{ij} = T_{ij} (\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$\text{Transponovaný tenzor: } \mathbf{T}^T = T_{ij} (\mathbf{e}_{ij})^T = T_{ij} \mathbf{e}_{ji} = T_{ij} (\vec{e}_j \otimes \vec{e}_i) = T_{ji} (\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

$$\text{Stopa tenzoru (skalár): } \text{tr } \mathbf{T} = T_{kk} = T_{11} + T_{22} + T_{33}$$

$$\text{Determinant: } \det \mathbf{T} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k} = \frac{1}{3!} \epsilon_{KLM} \epsilon_{klm} T_{kK} T_{lL} T_{mM} = \det \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

$$\text{Identická matice: } \mathbf{I} = \delta_{ij} \mathbf{e}_{ij} = \delta_{ij} (\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Inverzní matice: } \mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{I}, \quad (\mathbf{T}^{-1})_{ij} = \frac{\text{doplňěk } T_{ji}}{\det \mathbf{T}} \quad (\text{pouze je-li } \det \mathbf{T} \neq 0)$$

$$\text{Symetrická matice: } \mathbf{T} = \mathbf{T}^T$$

$$\text{Antisymetrická matice: } \mathbf{T} = -\mathbf{T}^T \quad (\text{je zřejmé, že } \text{tr } \mathbf{T} = 0)$$

$$\text{Ortogonální matice: } \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I} \quad (\Rightarrow \mathbf{Q}^T = \mathbf{Q}^{-1} \text{ a } |\det \mathbf{Q}| = 1; \det \mathbf{Q} = 1 \rightarrow \text{rotace, } \det \mathbf{Q} = -1 \rightarrow \text{zrcadlení})$$

$$\text{Rozklad tenzoru na symetrickou a antisymetrickou část: } \mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T)$$

$$\text{Rozklad tenzoru na izotropní část a deviator: } \mathbf{T} = \frac{1}{3}(\text{tr } \mathbf{T})\mathbf{I} + [\mathbf{T} - \frac{1}{3}(\text{tr } \mathbf{T})\mathbf{I}] = \left\{ \frac{1}{3}T_{kk}\delta_{ij} + [T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}] \right\} \mathbf{e}_{ij}$$

$$\mathbf{T} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} T_{11} + p & T_{12} & T_{13} \\ T_{21} & T_{22} + p & T_{23} \\ T_{31} & T_{32} & T_{33} + p \end{pmatrix}, \quad \text{kde } p = -\frac{1}{3}T_{kk}$$

$$\text{Rozklad tenzoru na izotropní část, antisymetrickou část a symetrickou bezstopou část:}$$

$$\mathbf{T} = \frac{1}{3}(\text{tr } \mathbf{T})\mathbf{I} + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T) + \frac{1}{2}[\mathbf{T} + \mathbf{T}^T - \frac{2}{3}(\text{tr } \mathbf{T})\mathbf{I}]$$

Násobení tenzorů a vektorů

$$\mathbf{T} \cdot \vec{v} = [T_{ij}(\vec{e}_i \otimes \vec{e}_j)] \cdot (v_k \vec{e}_k) = T_{ij}v_k[(\vec{e}_i \otimes \vec{e}_j) \cdot \vec{e}_k] = T_{ij}v_k\delta_{jk}\vec{e}_i = T_{ij}v_j\vec{e}_i = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3 \times 3} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{3 \times 1}$$

$$\vec{v} \cdot \mathbf{T} = (v_k \vec{e}_k) \cdot [T_{ij}(\vec{e}_i \otimes \vec{e}_j)] = T_{ij}v_k[\vec{e}_k \cdot (\vec{e}_i \otimes \vec{e}_j)] = T_{ij}v_k\delta_{ki}\vec{e}_j = T_{ij}v_i\vec{e}_j = T_{ji}v_j\vec{e}_i = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{1 \times 3} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3 \times 3}$$

$$\begin{aligned} \vec{v} \cdot \mathbf{T} \cdot \vec{u} &= (v_k \vec{e}_k) \cdot [T_{ij}(\vec{e}_i \otimes \vec{e}_j)] \cdot (v_l \vec{e}_l) = T_{ij}v_k u_l[\vec{e}_k \cdot (\vec{e}_i \otimes \vec{e}_j) \cdot \vec{e}_l] = T_{ij}v_k u_l \delta_{ki} \delta_{jl} = T_{ij}v_i u_j = \\ &= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{1 \times 3} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3 \times 3} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{3 \times 1} \end{aligned}$$

$$\begin{aligned} \mathbf{T} \cdot \mathbf{S} &= [T_{ij}(\vec{e}_i \otimes \vec{e}_j)] \cdot [S_{kl}(\vec{e}_k \otimes \vec{e}_l)] = T_{ij}S_{kl}[(\vec{e}_i \otimes \vec{e}_j) \cdot (\vec{e}_k \otimes \vec{e}_l)] = T_{ij}S_{kl}\delta_{jk}(\vec{e}_i \otimes \vec{e}_l) = T_{ij}S_{jl}(\vec{e}_i \otimes \vec{e}_l) = \\ &= T_{ij}S_{jl}\mathbf{e}_{il} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3 \times 3} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}_{3 \times 3} \end{aligned}$$

$$\mathbf{T} : \mathbf{S} = [T_{ij}(\vec{e}_i \otimes \vec{e}_j)] : [S_{kl}(\vec{e}_k \otimes \vec{e}_l)] = T_{ij}S_{kl}[(\vec{e}_i \otimes \vec{e}_j) : (\vec{e}_k \otimes \vec{e}_l)] = T_{ij}S_{kl}\delta_{jk}\delta_{il} = T_{ij}S_{ji}$$

Důležité vztahy a tvrzení:

$$(\vec{v} \otimes \vec{u}) \cdot \vec{w} = \vec{v}(\vec{u} \cdot \vec{w})$$

$$\vec{v} \cdot (\vec{u} \otimes \vec{w}) = (\vec{v} \cdot \vec{u})\vec{w}$$

$$(\vec{v} \otimes \vec{u}) \cdot (\vec{w} \otimes \vec{x}) = (\vec{u} \cdot \vec{w})(\vec{v} \otimes \vec{x})$$

$$\mathbf{I} \cdot \vec{v} = \vec{v} \cdot \mathbf{I} = \vec{v}$$

$$\vec{v} \cdot \mathbf{I} \cdot \vec{v} = v^2$$

$$\mathbf{T} \cdot \vec{v} = \vec{v} \cdot \mathbf{T}^T$$

$$\mathbf{T} \cdot (\phi \vec{v}) = \phi(\mathbf{T} \cdot \vec{v})$$

$$\mathbf{T} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{T} = \mathbf{T}$$

$$(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1} =: \mathbf{T}^{-T}$$

$$\vec{u} \cdot \mathbf{T} \cdot \vec{v} = \mathbf{T} : (\vec{v} \otimes \vec{u})$$

$$\vec{v} \cdot (\mathbf{A} \cdot \mathbf{B}) = (\vec{v} \cdot \mathbf{A}) \cdot \mathbf{B}$$

$$\mathbf{A} \cdot (\phi \mathbf{B}) = \phi(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$$

$$(\mathbf{A} \cdot \mathbf{B}) \cdot (\mathbf{C} \cdot \mathbf{D}) = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \cdot \mathbf{D}$$

$$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C} = (\mathbf{C} \cdot \mathbf{A}) : \mathbf{B}$$

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

$$(\mathbf{A}^T \cdot \mathbf{B}^T)^T = \mathbf{B} \cdot \mathbf{A}$$

$$\mathbf{A} \cdot \mathbf{A}^T = (\mathbf{A} \cdot \mathbf{A}^T)^T, \text{ tj. matice } \mathbf{A} \cdot \mathbf{A}^T \text{ je symetrická}$$

$$\mathbf{A}, \mathbf{B} \text{ jsou symetrické} \Leftrightarrow \mathbf{A} \cdot \mathbf{B} = (\mathbf{B} \cdot \mathbf{A})^T$$

$$\mathbf{P}, \mathbf{Q} \text{ jsou ortogonální} \Rightarrow \mathbf{P} \cdot \mathbf{Q} \text{ je rovněž ortogonální}$$

přehazování závorek
· VS X

$$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

$$\text{tr}(\mathbf{A} \cdot \mathbf{B}) = \text{tr}(\mathbf{B} \cdot \mathbf{A})$$

$$\det \mathbf{I} = 1$$

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det \mathbf{A} \det \mathbf{B}$$

$$\det(\phi \mathbf{T}) = \phi^3 \det \mathbf{T} \quad \text{pro } \mathbf{T}_{3 \times 3}$$

$$\det(\mathbf{A}^T) = \det \mathbf{A}$$

$$\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0 \quad \text{Jacobi}$$

$$(\vec{a} \times \vec{b}) + (\vec{c} \times \vec{d}) = (\vec{a} - \vec{c}) \times (\vec{b} - \vec{d}) + \vec{a} \times \vec{d} + \vec{c} \times \vec{b}$$

$$(\mathbf{T} \cdot \vec{a}) \times (\mathbf{T} \cdot \vec{b}) = (\det \mathbf{T}) \mathbf{T}^{-T} \cdot (\vec{a} \times \vec{b})$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = [\vec{a} \cdot (\vec{b} \times \vec{c})] \vec{a}$$

$$|\vec{a} \times \vec{b}|^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2 \quad \text{Lagrange}$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \quad \text{Binet-Cauchy}$$

Příklad: Ukažte, že $\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C}$

$$\begin{aligned}
 \mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) &= A_{ij}(\vec{e}_i \otimes \vec{e}_j) : [B_{kl}(\vec{e}_k \otimes \vec{e}_l) \cdot C_{mn}(\vec{e}_m \otimes \vec{e}_n)] = \\
 &= A_{ij}B_{kl}C_{mn}(\vec{e}_i \otimes \vec{e}_j) : [(\vec{e}_k \otimes \vec{e}_l) \cdot (\vec{e}_m \otimes \vec{e}_n)] = \\
 &= A_{ij}B_{kl}C_{mn}(\vec{e}_i \otimes \vec{e}_j) : (\delta_{lm}\vec{e}_k \otimes \vec{e}_n) = \\
 &= A_{ij}B_{kl}C_{mn}\delta_{lm}(\vec{e}_i \otimes \vec{e}_j) : (\vec{e}_k \otimes \vec{e}_n) = \\
 &= A_{ij}B_{kl}C_{mn}\delta_{lm}\delta_{jk}\delta_{in} = A_{ij}B_{jl}C_{li} = (\mathbf{A} \cdot \mathbf{B})_{il}C_{li} = \\
 &= (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C}
 \end{aligned}$$

Nebo ve zkrácené notaci:

$$\begin{aligned}
 \mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) &= A_{ij}(\mathbf{B} \cdot \mathbf{C})_{ji} = A_{ij}(B_{jk}C_{ki}) = (A_{ij}B_{jk})C_{ki} = \\
 &= (\mathbf{A} \cdot \mathbf{B})_{ik}C_{ki} = (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C}
 \end{aligned}$$

Příklad: Ukažte, že $(\vec{a} \times \vec{b})^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2$.

Ve zkrácené notaci:

$$\begin{aligned}
 (\vec{a} \times \vec{b})^2 &= (\vec{a} \times \vec{b})_i (\vec{a} \times \vec{b})_i = (\epsilon_{ijk} a_j b_k) (\epsilon_{imn} a_m b_n) = a_j b_k a_m b_n \epsilon_{ijk} \epsilon_{imn} = \\
 &= \boxed{a_j b_k a_m b_n (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km})} = a_j b_k a_m b_n \delta_{jm} \delta_{kn} - a_j b_k a_m b_n \delta_{jn} \delta_{km} = \\
 &= a_j b_k a_j b_k - a_j b_k a_k b_j = (a_j a_j) (b_k b_k) - (a_j b_j) (a_k b_k) = \\
 &= (\vec{a} \cdot \vec{a}) (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b}) (\vec{a} \cdot \vec{b}) = a^2 b^2 - (\vec{a} \cdot \vec{b})^2
 \end{aligned}$$

Příklad: Vyjádřete matici $\mathbf{A}$, pokud platí $(a\mathbf{B}) \cdot (\mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{M})^T = \mathbf{C}$.

$$(a\mathbf{B}) \cdot (\mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{M})^T = \mathbf{C}$$

$$(\mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{M})^T = a^{-1} \mathbf{B}^{-1} \cdot \mathbf{C}$$

$$\mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{M} = a^{-1} (\mathbf{B}^{-1} \cdot \mathbf{C})^T = a^{-1} \mathbf{C}^T \cdot \mathbf{B}^{-T}$$

$$\mathbf{A} \cdot \mathbf{M} = a^{-1} \mathbf{B}^{-T}$$

$$\mathbf{A} = a^{-1} \mathbf{B}^{-T} \cdot \mathbf{M}^{-1}$$

$$\begin{aligned}
 \text{Zkouška: } (a\mathbf{B}) \cdot (\mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{M})^T &= (a\mathbf{B}) \cdot (\mathbf{C}^T \cdot (a^{-1} \mathbf{B}^{-T} \cdot \mathbf{M}^{-1}) \cdot \mathbf{M})^T = \mathbf{B} \cdot (\mathbf{C}^T \cdot (\mathbf{B}^{-T} \cdot \mathbf{M}^{-1}) \cdot \mathbf{M})^T = \\
 &= \mathbf{B} \cdot ((\mathbf{C}^T \cdot \mathbf{B}^{-T}) \cdot (\mathbf{M}^{-1} \cdot \mathbf{M}))^T = \mathbf{B} \cdot (\mathbf{C}^T \cdot \mathbf{B}^{-T})^T = \mathbf{B} \cdot (\mathbf{B}^{-1} \cdot \mathbf{C}) = (\mathbf{B} \cdot \mathbf{B}^{-1}) \cdot \mathbf{C} = \mathbf{C}
 \end{aligned}$$

Vlastní čísla a vektory matice

$\mathbf{A} \cdot \vec{p} = \lambda \vec{p}$... λ , $\vec{p}$ označujeme jako vlastní čísla resp. vektory matice $\mathbf{A}$

$$\mathbf{A} \cdot \vec{p} - \lambda \vec{p} = (\mathbf{A} - \lambda \mathbf{I}) \cdot \vec{p} = 0$$

↑
charakteristická
matice

↓
charakteristická
rovnice

netriviální řešení $\Leftrightarrow \det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$-\lambda^3 + E_I \lambda^2 + E_{II} \lambda + E_{III} = 0 \quad \dots \text{charakteristický polynom}$$

$$E_I, E_{II}, E_{III} \quad \dots \text{invarianty matice } \mathbf{A}$$

Důležitá tvrzení:

1. Podobné matice mají stejný charakteristický polynom, a tedy stejná vlastní čísla.
2. Je-li $\mathbf{A}$ symetrická a reálná, pak její vlastní čísla jsou reálná. Je-li $\mathbf{A}$ navíc pozitivně definitní, pak jsou vlastní čísla kladná.
3. Necht' $\mathbf{A}$ je symetrická a reálná. Pak existuje taková ortogonální matice $\mathbf{Q}$, že matice $\mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$ je diagonální a reálná.

$\mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$ je Jordanův kanonický tvar matice $\mathbf{A}$:

$$\mathbf{J} = \mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q} \quad \dots \text{diagonální matice tvořená vlastními čísly}$$

$$\mathbf{Q} = (\vec{p}_1 \mid \vec{p}_2 \mid \vec{p}_3) \quad \dots \text{ortogonální matice tvořená vlastními vektory}$$

Matici $\mathbf{A}$, její determinant a odmocninu (pokud je $\mathbf{A}$ pozitivně definitní) lze pak vyjádřit jako:

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{J} \cdot \mathbf{Q}^T$$

$$\det \mathbf{A} = \det \mathbf{J}$$

$$\sqrt{\mathbf{A}} = \mathbf{Q} \cdot \sqrt{\mathbf{J}} \cdot \mathbf{Q}^T$$

Diferenciální operátor nábla:

$$\nabla = \vec{e}_k \frac{\partial}{\partial x_k}$$

$$\text{grad } a \equiv \nabla a = \vec{e}_k \frac{\partial a}{\partial x_k} = a_{,k} \vec{e}_k$$

$$\text{grad } \vec{v} \equiv \nabla \otimes \vec{v} = \vec{e}_k \frac{\partial}{\partial x_k} \otimes (v_l \vec{e}_l) = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \otimes \vec{e}_l) = \frac{\partial v_l}{\partial x_k} \mathbf{e}_{kl} = v_{l,k} \mathbf{e}_{kl} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$$(\text{grad } \vec{v})^T = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \otimes \vec{e}_l)^T = \frac{\partial v_l}{\partial x_k} (\vec{e}_l \otimes \vec{e}_k) = \frac{\partial v_l}{\partial x_k} \mathbf{e}_{lk} = \frac{\partial v_k}{\partial x_l} \mathbf{e}_{kl} = v_{k,l} \mathbf{e}_{kl} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$$\text{div } \vec{v} \equiv \nabla \cdot \vec{v} = \vec{e}_k \frac{\partial}{\partial x_k} \cdot (v_l \vec{e}_l) = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \cdot \vec{e}_l) = \frac{\partial v_l}{\partial x_k} \delta_{kl} = \frac{\partial v_k}{\partial x_k} = v_{k,k}$$

$$\begin{aligned} \text{div } \mathbf{T} \equiv \nabla \cdot \mathbf{T} &= \vec{e}_k \frac{\partial}{\partial x_k} \cdot [T_{lm} (\vec{e}_l \otimes \vec{e}_m)] = \frac{\partial T_{lm}}{\partial x_k} [\vec{e}_k \cdot (\vec{e}_l \otimes \vec{e}_m)] = \frac{\partial T_{lm}}{\partial x_k} \delta_{kl} \vec{e}_m = \frac{\partial T_{lm}}{\partial x_l} \vec{e}_m = T_{lm,l} \vec{e}_m \\ &= \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix}_{1 \times 3} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3 \times 3} \end{aligned}$$

Diferenciální operátor nabra – užitečné identity:

$$\nabla(ab) = a\nabla b + b\nabla a$$

$$\nabla \cdot (a\vec{v}) = a\nabla \cdot \vec{v} + \vec{v} \cdot \nabla a$$

$$\nabla \times (a\vec{v}) = a\nabla \times \vec{v} + \nabla a \times \vec{v}$$

$$\nabla(\vec{u} \cdot \vec{v}) = \nabla \vec{u} \cdot \vec{v} + \nabla \vec{v} \cdot \vec{u}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla \times (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v} + \vec{u} \nabla \cdot \vec{v} - \vec{v} \nabla \cdot \vec{u}$$

$$\vec{u} \times (\nabla \times \vec{v}) = \nabla \vec{v} \cdot \vec{u} - \vec{u} \cdot \nabla \vec{v}$$

$$\nabla \times (\nabla a) = 0$$

$$\nabla \cdot (\nabla \times \vec{v}) = 0$$

$$\nabla(\nabla \cdot \vec{v}) = \nabla \cdot [(\nabla \vec{v})^T]$$

$$\nabla \cdot (\nabla \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla \times (\nabla \times \vec{v})$$

$$\nabla(a\vec{v}) = a\nabla \vec{v} + \nabla a \otimes \vec{v}$$

$$\nabla \cdot (a\mathbf{T}) = a\nabla \cdot \mathbf{T} + \nabla a \cdot \mathbf{T}$$

$$\nabla \cdot (\vec{u} \otimes \vec{v}) = \vec{v} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{v}$$

K odvození potřebujeme vztahy pro operace s vektory a tenzory a pravidla pro derivování funkcí.

Příklad:

$$\nabla \cdot (a\mathbf{T}) = (\vec{e}_k \frac{\partial}{\partial x_k}) \cdot [a T_{lm} (\vec{e}_l \otimes \vec{e}_m)] = [\frac{\partial}{\partial x_k} (a T_{lm})] [\vec{e}_k \cdot (\vec{e}_l \otimes \vec{e}_m)] = [T_{lm} \frac{\partial a}{\partial x_k} + a \frac{\partial T_{lm}}{\partial x_k}] [\vec{e}_k \cdot (\vec{e}_l \otimes \vec{e}_m)] = [T_{lm} \frac{\partial a}{\partial x_k} + a \frac{\partial T_{lm}}{\partial x_k}] \delta_{kl} \vec{e}_m =$$

↑
 ∇

↑
 $a\mathbf{T}$

↑
oddělíme derivování
a vektorové operace

↑
derivace
součinu

↑
provedeme
vektorové
operace

$$= [T_{lm} \frac{\partial a}{\partial x_l} + a \frac{\partial T_{lm}}{\partial x_l}] \vec{e}_m = \frac{\partial a}{\partial x_l} T_{lm} \vec{e}_m + a \frac{\partial T_{lm}}{\partial x_l} \vec{e}_m = \nabla a \cdot \mathbf{T} + a \nabla \cdot \mathbf{T}$$

↑
 $\nabla a \cdot \mathbf{T}$

↑
 $a \nabla \cdot \mathbf{T}$

Příklad: Ukažte, že $\nabla(a\vec{v}) = a\nabla\vec{v} + \nabla a \otimes \vec{v}$.

$$\begin{aligned}\nabla(a\vec{v}) &= \vec{e}_i \frac{\partial}{\partial x_i} \otimes (av_j \vec{e}_j) = \frac{\partial(av_j)}{\partial x_i} (\vec{e}_i \otimes \vec{e}_j) = \frac{\partial a}{\partial x_i} v_j (\vec{e}_i \otimes \vec{e}_j) + a \frac{\partial v_j}{\partial x_i} (\vec{e}_i \otimes \vec{e}_j) = \\ &= \left(\frac{\partial a}{\partial x_i} \vec{e}_i \right) \otimes (v_j \vec{e}_j) + a \vec{e}_i \frac{\partial}{\partial x_i} \otimes (v_j \vec{e}_j) = \nabla a \otimes \vec{v} + a \nabla \vec{v}.\end{aligned}$$

Ve zkrácené notaci:

$$[\nabla(a\vec{v})]_{ij} = \frac{\partial}{\partial x_i} (av_j) = \frac{\partial a}{\partial x_i} v_j + a \frac{\partial v_j}{\partial x_i} = (\nabla a)_i v_j + a (\nabla \vec{v})_{ij} = (\nabla a \otimes \vec{v})_{ij} + a \nabla(\vec{v})_{ij}$$

Příklad: Ukažte, že $\vec{u} \times (\nabla \times \vec{v}) = \nabla \vec{v} \cdot \vec{u} - \vec{u} \cdot \nabla \vec{v}$.

$$\begin{aligned}[\vec{u} \times (\nabla \times \vec{v})]_i &= \epsilon_{ijk} u_j (\nabla \times \vec{v})_k = \epsilon_{ijk} u_j \left(\epsilon_{klm} \frac{\partial v_m}{\partial x_l} \right) = \epsilon_{ijk} \epsilon_{klm} u_j \frac{\partial v_m}{\partial x_l} = \\ &= \epsilon_{kij} \epsilon_{klm} u_j \frac{\partial v_m}{\partial x_l} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) u_j \frac{\partial v_m}{\partial x_l} = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j} = \\ &= [\nabla \vec{v} \cdot \vec{u}]_i - [\vec{u} \cdot \nabla \vec{v}]_i\end{aligned}$$

Nezapomeňte, že při aplikaci Kroneckerovy delty musíte podržet index i .

Příklad: Ukažte, že $\nabla(\nabla \cdot \vec{v}) = \nabla \cdot [(\nabla \vec{v})^T]$.

$$[\nabla(\nabla \cdot \vec{v})]_i = \frac{\partial}{\partial x_i} \frac{\partial v_j}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_j} (\nabla \vec{v})_{ij} = \frac{\partial}{\partial x_j} (\nabla \vec{v})_{ji}^T = [\nabla \cdot (\nabla \vec{v})^T]_i$$

Příklad: Vyjádřete $\nabla \times \nabla \times \vec{v}$ bez použití vektorového součinu.

$$\begin{aligned} [\nabla \times \nabla \times \vec{v}]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \vec{v})_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\epsilon_{klm} \frac{\partial v_m}{\partial x_l} \right) = \\ &= \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial v_m}{\partial x_l} = \epsilon_{kij} \epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial v_m}{\partial x_l} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial v_m}{\partial x_l} = \\ &= \frac{\partial}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_i} \frac{\partial v_j}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} = \\ &= \frac{\partial}{\partial x_i} (\nabla \cdot \vec{v}) - \frac{\partial}{\partial x_j} (\nabla \vec{v})_{ji} = [\nabla \nabla \cdot \vec{v}]_i - [\nabla \cdot \nabla \vec{v}]_i \end{aligned}$$

$$\nabla \times \nabla \times \vec{v} = \nabla \nabla \cdot \vec{v} - \nabla \cdot \nabla \vec{v}$$