

mization” procedure. It is based on the work of Rajagopal & Srinivasa (2004) and summarized in Málek & Průša (2016). The procedure consists of *a priori* stating how the model material dissipates energy (i.e. produces entropy). Within this approach, the constitutive law is obtained by assuming that the body always deforms such as to maximize its entropy production while simultaneously complying with the thermodynamical constraints imposed by fundamental balance laws.

In Section 1.3 we briefly review how Maxwell viscoelastic fluid can be derived within such framework. Similarly as in the traditional approach, the underlying idea is to assume the total deformation of a viscoelastic body to be composed of two parts, one being dissipative and the other elastic. A special configuration of the considered body is introduced to distinguish these: so-called natural configuration. The natural configuration, respectively its temporal evolution, corresponds only to the dissipative part of the total deformation. The entropy production maximization technique is very different from the traditional one, but the principle of material frame-indifference is equally important within its concept. It is because the principle has to be applied when proposing a model’s entropy production rate. With natural configuration involved, the proper application of PMFI is a delicate exercise, which has not been to our knowledge discussed in the literature. We pay attention to it in Section 1.3.

1.1 Material frame-indifference

In this section we adopt the conceptual approach of Martinec (2011) (see chapter 5. therein) and expand it by considering non-Cartesian frames. Such expansion will be needed in Section 1.2. Regarding key terminology we strictly follow the analysis by Liu & Sampaio (2014), who distinguish between the concepts of *frame indifference*, *postulate of Euclidean objectivity*, *principle of material frame-indifference*, and *condition of material objectivity*.

Let a body $\mathcal{B}$ be a set of material points. Motion of the body will be defined as a mapping of the body, at each time $t \in [0, \infty)$, into the Euclidean space $\mathbb{E}$,

$$\chi : \mathcal{B} \times [0, \infty) \rightarrow \mathbb{E}. \quad (1.3)$$

We will consider two observers of the body. Each observer can be understood as a *frame*, the frame allowing the observer to measure positions, distances and angles in the Euclidean space $\mathbb{E}$. Frame can be formalized as a mapping that, at a given time t , associates each point of the Euclidean space with its coordinates in the frame $\phi : \mathbb{E} \times [0, \infty) \rightarrow \mathbb{R}^3$. Motion of the body, when measured in frame ϕ , is defined as the composite mapping $\chi_\phi = \phi \circ \chi$,

$$\chi_\phi : \mathcal{B} \times [0, \infty) \rightarrow \mathbb{R}^3, \quad (1.4)$$

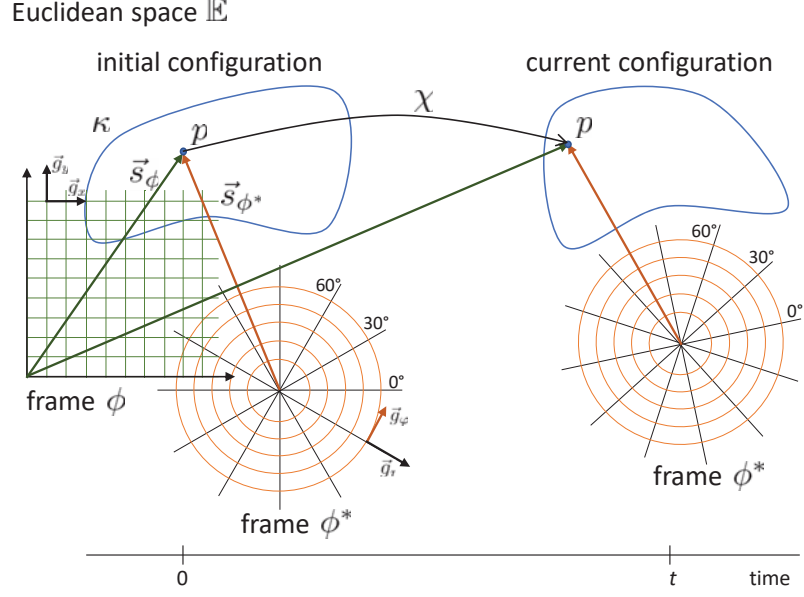


Figure 1.2: Position of particle p at times 0 and t , as viewed from fixed Cartesian frame $\phi: \mathbb{E} \rightarrow \mathbb{R}^3$ and from moving polar frame $\phi^*: \mathbb{E} \times [0, \infty) \rightarrow \mathbb{R}^3$.

where the triplets of real numbers denote the coordinates of material points' positions at time t in frame ϕ . During any motion χ , measured by two observers ϕ and ϕ^* , we can identify the position of each particle p by its coordinates $\mathbf{x} = \chi_\phi(p, t)$ in frame ϕ and its coordinates $\mathbf{x}^* = \chi_{\phi^*}(p, t)$ in frame ϕ^* . The time dependent mapping between coordinates of the same point in the Euclidean space $\mathbb{E}$ will be referred to as *change of frame*: $\mathbf{x}^* = \mathbf{x}^*(\mathbf{x}, t)$, resp $\mathbf{x} = \mathbf{x}(\mathbf{x}^*, t)$. See Appendix 1.A for a worked out example.

Each frame contains an *origin*, the point with zero coordinates, and a local basis formed by vectors tangent to coordinate lines at each point (see Fig. 1.2). The local basis $\vec{e}_{i=1,2,3}$ of frame ϕ is related with the local basis $\vec{g}_{i=1,2,3}$ of frame ϕ^* via

$$\vec{g}_j(\mathbf{x}^*, t) = \frac{\partial x^i(\mathbf{x}^*, t)}{\partial x_*^j} \vec{e}_i(\mathbf{x}, t) =: Q_j^i(\mathbf{x}^*, t) \vec{e}_i(\mathbf{x}, t), \quad (1.5)$$

where we have defined the shifter tensor Q_j^i and used the Einstein summation convention. The local vector bases $\vec{e}_i$ and $\vec{g}_i$ are but two different bases of the same vector space $\mathbb{V}$, which is the translational space of the Euclidean space $\mathbb{E}$.

Let us assume that each material point, at any time t , holds a scalar quantity $a: \mathcal{B} \times [0, \infty) \rightarrow \mathbb{R}$. The functions $a_\phi: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$ and $a_{\phi^*}: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$, defined as $a_\phi(\chi_\phi(p, t), t) := a(p, t)$, resp. $a_{\phi^*}(\chi_{\phi^*}(p, t), t) := a(p, t)$, are representations of the scalar field a in the frames ϕ and ϕ^* . Clearly, $a_{\phi^*}(\mathbf{x}^*, t) = a_\phi(\mathbf{x}, t)$. We say that the scalar field a is *frame indifferent*, or *objective*, because for each point in the Euclidean space $\mathbb{E}$ its value is given, irrespective of the chosen frame.

Similarly, for any vector quantity $\vec{a}$ belonging to $\mathbb{V}$ we say that it is objective if $\vec{a}_{\phi^*}(\mathbf{x}^*, t) = \vec{a}_\phi(\mathbf{x}, t)$. For the components of such vector field with respect to

the local bases of $\vec{e}_i$ and $\vec{g}_j$ eq. (1.5) gives

$$\begin{aligned}\vec{a}_{\phi^*}(\mathbf{x}^*, t) &= a_{\phi^*}^j(\mathbf{x}^*, t) \vec{g}_j(\mathbf{x}^*, t) = a_{\phi^*}^j(\mathbf{x}^*, t) Q_j^i(\mathbf{x}^*, t) \vec{e}_i(\mathbf{x}, t) \\ &= \vec{a}_{\phi}(\mathbf{x}, t) = a_{\phi}^i(\mathbf{x}, t) \vec{e}_i(\mathbf{x}, t),\end{aligned}$$

which provides the transformation properties of components of objective vectors, $a_{\phi}^i(\mathbf{x}, t) = Q_j^i(\mathbf{x}^*, t) a_{\phi^*}^j(\mathbf{x}^*, t)$.

For any second-order contravariant tensor $\vec{a}$ belonging to $\mathbb{V} \otimes \mathbb{V}$ we get, following the same procedure,

$$\vec{a}_{\phi^*} = a_{\phi^*}^{kl} \vec{g}_k \otimes \vec{g}_l = a_{\phi^*}^{kl} (Q_k^i \vec{e}_i) \otimes (Q_l^j \vec{e}_j) = a_{\phi^*}^{kl} Q_k^i Q_l^j \vec{e}_i \otimes \vec{e}_j = \vec{a}_{\phi} = a_{\phi}^{ij} \vec{e}_i \otimes \vec{e}_j,$$

that its components transform as

$$a_{\phi}^{ij}(\mathbf{x}, t) = Q_k^i(\mathbf{x}^*, t) Q_l^j(\mathbf{x}^*, t) a_{\phi^*}^{kl}(\mathbf{x}^*, t). \quad (1.6)$$

The basis of the space of all linear mappings $\mathbb{V} \rightarrow \mathbb{R}$, denoted as $\mathcal{L}(\mathbb{V})$, will be referred to as the local dual basis. It is formed by one-forms $\underline{e}^i$, resp. $\underline{g}^i$, which are defined through the relations

$$\underline{e}^i(\vec{e}_j) = \delta_j^i; \quad \underline{g}^i(\vec{g}_j) = \delta_j^i, \quad (1.7)$$

where δ_j^i is the Kronecker delta. In virtue of the identity

$$\delta_j^i = \frac{\partial x_*^i(\mathbf{x}, t)}{\partial x^k} \frac{\partial x^k(\mathbf{x}^*, t)}{\partial x_*^j} \quad (1.8)$$

we see that (insert for δ_j^i in the second relation of eq. (1.7) and expand $\vec{g}_j$ on its left-hand side – the following then stems from the linearity of one-forms)

$$\underline{g}^i(\mathbf{x}^*, t) = \frac{\partial x_*^i(\mathbf{x}, t)}{\partial x^j} \underline{e}^j(\mathbf{x}, t) = \{Q^{-1}(\mathbf{x}, t)\}_j^i \underline{e}^j(\mathbf{x}, t), \quad (1.9)$$

where $Q^{-1}(\mathbf{x}, t)$ is the inverse shifter tensor, $Q_k^i(\mathbf{x}^*, t) \{Q^{-1}(\mathbf{x}, t)\}_j^k = \delta_j^i$. The transformation properties of one-forms, or covectors, are thus inverse when compared to those of vectors. Typical example of a one-form is the gradient of a scalar field:

$$\nabla a = \frac{\partial a_{\phi}(\mathbf{x}, t)}{\partial x^i} \underline{e}^i(\mathbf{x}, t) = \frac{\partial a_{\phi^*}(\mathbf{x}^*, t)}{\partial x_*^k} \frac{\partial x_*^k(\mathbf{x}, t)}{\partial x^i} \underline{e}^i(\mathbf{x}, t) = \frac{\partial a_{\phi^*}(\mathbf{x}^*, t)}{\partial x_*^k} \underline{g}^k(\mathbf{x}^*, t),$$

where we have used the chain rule in differentiating $a_{\phi}(\mathbf{x}, t) = a_{\phi^*}(\mathbf{x}^*(\mathbf{x}, t), t)$ to show that the components of the gradient indeed transform according to eq. (1.9), that is, $\{\nabla a_{\phi}(\mathbf{x}, t)\}_i = \{Q^{-1}\}_i^k(\mathbf{x}^*, t) \{\nabla a_{\phi^*}(\mathbf{x}^*, t)\}_k$.

Performing the above procedure for components of an objective vector field $\vec{a}$

yields the transformation properties of the gradient of objective vector field:

$$\{\nabla \vec{a}_\phi(\mathbf{x}, t)\}^i_j = Q^i_k(\mathbf{x}, t) \{Q^{-1}(\mathbf{x}^*, t)\}^l_j \{\nabla \vec{a}_{\phi^*}(\mathbf{x}^*, t)\}^k_l, \quad (1.10)$$

where $\{\nabla \vec{a}_\phi(\mathbf{x}, t)\}^i_j := \partial a_\phi^i / \partial x^j$. The shifter tensor in eq. (1.10) comes from the objectivity of the vector field, $a_\phi^i(\mathbf{x}, t) = Q^i_j a_{\phi^*}^j(\mathbf{x}^*(\mathbf{x}, t), t)$, and the inverse shifter tensor stems from the differentiation. Eq. (1.10) is equivalent to saying that the gradient of objective vector field is a second-order, right co-variant mixed tensor (contravariant in its first index, covariant in the second index). Such tensors can be written as a linear combination of $\vec{e}_i \otimes \underline{e}^j$, so that $\nabla \vec{a}_\phi(\mathbf{x}, t) = \{\nabla \vec{a}_\phi(\mathbf{x}, t)\}^i_j \vec{e}_i \otimes \underline{e}^j$.

Not all vectors are objective. Typical examples of frame dependent vector fields are the velocity and the acceleration of a body. Velocity of a particle is the rate of change of its position vector. We will denote the position vector of particle p in frame ϕ as $\vec{s}_\phi(\mathbf{x}, t)$, resp. $\vec{s}_{\phi^*}(\mathbf{x}^*, t)$ in frame ϕ^* , where $\mathbf{x} = \chi_\phi(p, t)$ and $\mathbf{x}^* = \chi_{\phi^*}(p, t)$ are the respective coordinates of point p . The position vectors are related via

$$\vec{s}_\phi(\mathbf{x}, t) = \vec{s}_{\phi^*}(\mathbf{x}^*, t) + \vec{c}(t), \quad (1.11)$$

where $\vec{c}(t)$ is a vector connecting the origins of both frames. To see how the velocity of a particle, when measured in frame ϕ , is related to the velocity observed in frame ϕ^* , we must apply the material derivative $\frac{\mathcal{D}}{\mathcal{D}t} := \frac{\partial}{\partial t} \Big|_{p=\text{const}}$ to eq. (1.11). First, let us express the vectors in eq. (1.11) in terms of its components in the corresponding local vector bases

$$s_\phi^i(\mathbf{x}, t) \vec{e}_i(\mathbf{x}, t) = s_{\phi^*}^j(\mathbf{x}^*, t) \vec{g}_j(\mathbf{x}^*, t) + \vec{c}(t). \quad (1.12)$$

Though generally not necessary, it is usually instructive to have one observer equipped with a simple Cartesian frame attached to the Euclidean space, $\phi: \mathbb{E} \rightarrow \mathbb{R}^3$, with its basis being identical at every point of the Euclidean space, $\vec{e}_i(\mathbf{x}, t) = \vec{e}_i$, and experiment only with the other observer $\phi^*: \mathbb{E} \times [0, \infty) \rightarrow \mathbb{R}^3$. Eq. (1.12) then reads

$$x^i(p, t) \vec{e}_i = s_{\phi^*}^j(\mathbf{x}^*(p, t), t) \vec{g}_j(\mathbf{x}^*(p, t), t) + c_\phi^k(t) \vec{e}_k, \quad (1.13)$$

because in the Cartesian frame components of the position vector are equal to the coordinates of the respective point, $s_\phi^i(\mathbf{x}(p, t), t) = x^i(p, t)$. By applying material derivative to eq. (1.13) we get

$$v_\phi^i \vec{e}_i = \frac{\mathcal{D} s_{\phi^*}^j}{\mathcal{D}t} \vec{g}_j + s_{\phi^*}^j \frac{\partial \vec{g}_j}{\partial x_*^l} \frac{\mathcal{D} x_*^l}{\mathcal{D}t} + s_{\phi^*}^j \frac{\partial \vec{g}_j}{\partial t} + \frac{\mathcal{D} c_\phi^k}{\mathcal{D}t} \vec{e}_k, \quad (1.14)$$

where the particle's velocity in the Cartesian frame was identified, $v_\phi^i := (\mathcal{D} x^i) / (\mathcal{D}t)$. The first two terms on the right-hand side of eq. (1.14) together form the velocity $\vec{v}_{\phi^*}$, with the second term correcting for the change of the local basis along the

trajectory of the particle (first term alone would equal the velocity $\vec{v}_{\phi^*}$ if the local basis of ϕ^* was spatially homogeneous). The third term is related to the relative motion of both frames (see below). The last term describes the relative velocity of the origins of both frames and does not depend on which particle p we are tracing.

By *Euclidean change of frame* we will understand a change of frame that satisfies

$$s_{\phi^*}^j(\mathbf{x}^*, t) \frac{\partial \vec{g}_j(\mathbf{x}^*, t)}{\partial t} = \vec{\omega}(t) \times \vec{s}_{\phi^*}(\mathbf{x}^*, t) \quad (1.15)$$

for all points $\mathbf{x}^*$ and times t . The vector $\vec{\omega}(t)$ is the relative rotation of both frames and it is dependent on time only. It can be related with the tensor

$$\Omega_j^i(\mathbf{x}^*, t) := \frac{\partial Q_k^i(\mathbf{x}^*, t)}{\partial t} \{Q^{-1}(\mathbf{x}(\mathbf{x}^*, t), t)\}_j^k, \quad (1.16)$$

through the relation $\Omega_j^i a^j = \xi_{ijk} \omega^j a^k$, valid for any vector $\vec{a}$. ξ_{ijk} denotes the Levi-Civita symbol. In other words, the change of frame is Euclidean if and only if $\Omega_j^i(\mathbf{x}^*, t)$ is antisymmetric and spatially homogeneous. In Appendix 1.A the formulae (1.14) are worked out for a particular example, explaining the meaning of each term in eq. (1.14) and discussing the properties of $\Omega_j^i(\mathbf{x}^*, t)$ for Euclidean changes of frame, for which the relative motion of frames ϕ and ϕ^* is a combination of translation and rotation only.

Note that it would not have been possible to distinguish between Euclidean and other changes of frame, had we, in the beginning, restricted ourselves only to Cartesian frames (with equal metric). Cartesian frames with equal metric are always only translated and rotated with respect to each other. For the rest of this section, however, we will consider Euclidean changes of frame only and for brevity we will thus abandon curvilinear coordinates (we return to them in Section 1.2).

For Cartesian coordinates one gets $\vec{g}_j(\mathbf{x}^*, t) = \vec{g}_j(t)$, $s_{\phi^*}^j(\mathbf{x}^*(p, t), t) = x_*^j(p, t)$, $Q_j^i(\mathbf{x}^*, t) = Q_j^i(t)$ and eq. (1.14) reads

$$v_{\phi}^i(\mathbf{x}, t) \vec{e}_i = v_{\phi^*}^j(\mathbf{x}^*, t) \vec{g}_j(t) + \xi_{ijk} \omega_j(t) x_*^k(p, t) \vec{g}_i(t) + \frac{dc_{\phi}^k(t)}{dt} \vec{e}_k. \quad (1.17)$$

Applying material derivative again gives a relation between the accelerations of the particle, as viewed by the two observers:

$$a_{\phi}^i \vec{e}_i = a_{\phi^*}^j \vec{g}_j + v_{\phi^*}^j \frac{\partial \vec{g}_j}{\partial t} + \xi_{ijk} \frac{d\omega_j}{dt} x_*^k \vec{g}_i + \xi_{ijk} \omega_j \frac{\mathcal{D}x_*^k}{\mathcal{D}t} \vec{g}_i + \xi_{ijk} \omega_j x_*^k \frac{\partial \vec{g}_i}{\partial t} + \frac{d^2 c_{\phi}^k}{(dt)^2} \vec{e}_k,$$

where $a_{\phi}^i := \mathcal{D}v_{\phi}^i / \mathcal{D}t$, resp. $a_{\phi^*}^i := \mathcal{D}v_{\phi^*}^i / \mathcal{D}t$. Since eq. (1.15) is valid for any position vector, it must be valid for any vector in general, and we can rewrite the second, fourth and fifth terms as cross products:

$$a_{\phi}^i \vec{e}_i = a_{\phi^*}^j \vec{g}_j + \vec{\omega} \times \vec{v}_{\phi^*} + \xi_{ijk} \frac{d\omega_j}{dt} x_*^k \vec{g}_i + \vec{\omega} \times \vec{v}_{\phi^*} + \vec{\omega} \times (\vec{\omega} \times \vec{s}_{\phi^*}) + \frac{d^2 c_{\phi}^k}{(dt)^2} \vec{e}_k. \quad (1.18)$$

In direct notation, that is, without expanding the vectors into local bases, eq. (1.18) reads

$$\vec{a}_\phi = \vec{a}_{\phi^*} + 2\vec{\omega} \times \vec{v}_{\phi^*} + \frac{d\vec{\omega}}{dt} \times \vec{s}_{\phi^*} + \vec{\omega} \times (\vec{\omega} \times \vec{s}_{\phi^*}) + \frac{d^2 \vec{c}_\phi}{(dt)^2}, \quad (1.19)$$

which is valid for any Euclidean change of frame. We see that the accelerations in frames ϕ and ϕ^* differ by the well known Coriolis acceleration, Euler acceleration, centrifugal acceleration, and by the relative acceleration of the frames' origins.

Once we have derived the transformation properties of objective vectors, one-forms and contravariant tensors, as well as the transformation properties of material particle's velocity and acceleration, we can discuss the transformation properties of the equation of motion. Let us assume that frame ϕ is an inertial frame. The equation of motion then acquires the form

$$\nabla \cdot \overleftarrow{\tau}_\phi + \vec{f}_\phi = \rho_\phi \vec{a}_\phi. \quad (1.20)$$

To find the equation of motion in some non-inertial frame ϕ^* , one must find the transformation properties of the density ρ_ϕ , the volumetric body force $\vec{f}_\phi$, and the Cauchy stress tensor $\overleftarrow{\tau}_\phi$. We agree with Liu & Sampaio (2014), that “unlike the acceleration, transformation properties of non-kinematic quantities cannot be deduced theoretically”. Instead, the transformation properties of non-kinematic quantities must be postulated. The *postulate of Euclidean objectivity* says that, for any Euclidean change of frame, the density, the body force and the Cauchy stress tensor are frame indifferent, and so their components transform in the following manner:

$$\begin{aligned} \rho_\phi(\mathbf{x}, t) &= \rho_{\phi^*}(\mathbf{x}^*, t); & f_\phi^i(\mathbf{x}, t) &= Q^i_j(t) f_{\phi^*}^j(\mathbf{x}^*, t); \\ \tau_\phi^{ij}(\mathbf{x}, t) &= Q^i_k(t) Q^j_l(t) \tau_{\phi^*}^{kl}(\mathbf{x}^*, t). \end{aligned} \quad (1.21)$$

In other words, these quantities are objective scalar, vector, and contravariant tensor fields respectively. The equation of motion in a non-inertial frame ϕ^* can then be written as

$$\nabla \cdot \overleftarrow{\tau}_{\phi^*} + \vec{f}_{\phi^*} = \rho_{\phi^*} \vec{a}_{\phi^*} + \rho_{\phi^*} \vec{i}_{\phi^*}, \quad (1.22)$$

where $\rho_{\phi^*} \vec{i}_{\phi^*}$ are the so-called fictitious forces, $\vec{i}_{\phi^*}$ being the accelerations named below eq. (1.19).

The primary focus of this chapter is on the properties of constitutive equations for the Cauchy stress tensor. We will assume a body whose stress state is determined only by the body's motion,

$$\tau_\phi^{ij}(\mathbf{x}(p, t), t) = \mathcal{F}_\phi^{ij}(\chi_\phi; p), \quad (1.23)$$

where $\mathcal{F}_\phi^{ij}$ is a functional that can generally depend on the entire history of the motion that the body has experienced. From the postulate of Euclidean objectivity it immediately follows that

$$\mathcal{F}_\phi^{ij}(\chi_\phi; p) = Q_k^i(t) Q_l^j(t) \mathcal{F}_{\phi^*}^{kl}(\chi_{\phi^*}; p) = Q_k^i(t) Q_l^j(t) \mathcal{F}_{\phi^*}^{kl}(*(\chi_\phi); p), \quad (1.24)$$

where $*$ denotes the Euclidean change of frame $\mathbf{x}^* = \mathbf{x}^*(\mathbf{x}, t)$ of the entire history of the motion χ_ϕ . Eq. (1.24) does not yet impose any restrictions on the constitutive functional $\mathcal{F}_\phi^{ij}$ itself. It is a relation between two constitutive functionals $\mathcal{F}_\phi^{ij}$ and $\mathcal{F}_{\phi^*}^{ij}$, defining the Cauchy stress tensor for two different observers. The desired restriction comes when eq. (1.24) is combined with the *principle of material frame-indifference*, which says that the form of the constitutive equation does not depend on observer:

$$\mathcal{F}_\phi^{ij}(\bullet; p) = \mathcal{F}_{\phi^*}^{ij}(\bullet; p). \quad (1.25)$$

Only then one arrives at the *condition of material objectivity*:

$$\mathcal{F}_\phi^{ij}(\chi_\phi; p) = Q_k^i(t) Q_l^j(t) \mathcal{F}_\phi^{kl}(*(\chi_\phi); p), \quad (1.26)$$

where $*$ stands for any Euclidean change of frame. Eq. (1.26) is a fundamental result in constitutive theories. It is used for testing whether a suggested constitutive law is physically consistent. An intuitive way to interpret the condition of material objectivity is to consider χ_ϕ and $*(\chi_\phi)$ to be two different motions of the same body, these motions being measured in the frame ϕ . The deformation experienced by the body in these motions is, however, identical, because the motions differ only by a rigid body motion (see the definition of Euclidean changes of frame). The condition (1.26) thus says that the stress state of the material depends on the experienced deformation only.

Note that if we wanted to stay within the general framework of curvilinear coordinates, we could not write eq. (1.25)². Indeed, in some textbooks it is directly eq. (1.26) that is postulated, referred to as the “principle of invariance under superimposed rigid body motions”. It is because eq. (1.26) is valid for any frame, including curvilinear ones, as long as $*(\chi_\phi)$ is a motion composed of the motion χ_ϕ and some rigid body translation and rotation, measured in the frame ϕ . The shifter tensor in eq. (1.26) is, however, spatially dependent for curvilinear frames.

So far we have worked with motions defined as mappings of the body $\mathcal{B}$. The domain of such mappings was a set of material points. In real applications, this is rarely the case. Motions are usually defined with respect to some reference configuration. By a *reference configuration* of the body $\mathcal{B}$ in frame ϕ we understand a mapping $\kappa_\phi: \mathcal{B} \rightarrow \mathbb{R}^3$, which is a time independent mapping that associates each material point uniquely with a set of coordinates in the frame ϕ . Reference con-

²See the example in Appendix 1.A, where eq. (1.25) is not valid because frame ϕ is Cartesian and frame ϕ^* is polar.

figuration is often chosen as the initial configuration the body, $\kappa_\phi(\bullet) = \chi_\phi(\bullet, 0)$, but not necessarily. For example, when modelling small deformations of a planet, the reference configuration is usually taken as a sphere, even though the planet never was a sphere during the studied motion (see e.g. the modelling study in Chapter 2).

When κ_ϕ is used to replace the set of material points $\mathcal{B}$ by $\mathbb{R}^3$ in the definition of motion χ_ϕ , we get the traditional notion of a motion, used in practical applications:

$$\chi_{\kappa,\phi} : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3, \quad (1.27)$$

where $\chi_{\kappa,\phi}$ is the motion χ_ϕ with respect to the reference configuration κ_ϕ . The coordinates of material point p in the reference configuration κ_ϕ are usually denoted as $\mathbf{X} = \kappa_\phi(p)$. The coordinates of such particle throughout the motion are obtained as $\mathbf{x} = \chi_{\kappa,\phi}(\mathbf{X}, t)$. Distinguishing between χ_ϕ and $\chi_{\kappa,\phi}$ has important implications for constitutive theory, which was pointed out by Liu & Sampaio (2014). Constitutive functional $\mathcal{F}_\phi^{ij}$, defined for motions χ_ϕ , is not ideal for real applications. Instead, it is convenient to introduce a constitutive functional $\mathcal{F}_{\kappa,\phi}^{ij}$, defined on motions in referential description:

$$\mathcal{F}_\phi^{ij}(\chi_\phi; p) = \mathcal{F}_\phi^{ij}(\chi_{\kappa,\phi} \circ \kappa_\phi; \kappa_\phi^{-1}(\mathbf{X})) =: \mathcal{F}_{\kappa,\phi}^{ij}(\chi_{\kappa,\phi}; \mathbf{X}). \quad (1.28)$$

The principle of material frame-indifference (1.25), when motion χ_ϕ is inserted in its both sides, implies that

$$\begin{aligned} \mathcal{F}_{\kappa,\phi}^{ij}(\chi_{\kappa,\phi}; \mathbf{X}) &= \mathcal{F}_\phi^{ij}(\chi_\phi; p) \stackrel{(1.25)}{=} \mathcal{F}_{\phi^*}^{ij}(\chi_\phi; p) = \mathcal{F}_{\phi^*}^{ij}(\chi_{\kappa,\phi} \circ \kappa_\phi; \kappa_\phi^{-1}(\mathbf{X})) \\ &= \mathcal{F}_{\phi^*}^{ij}(\chi_{\kappa,\phi} \circ \kappa_\phi \circ \kappa_{\phi^*}^{-1} \circ \kappa_{\phi^*}; \kappa_{\phi^*}^{-1} \circ \kappa_{\phi^*} \circ \kappa_\phi^{-1}(\mathbf{X})) = \mathcal{F}_{\kappa^*,\phi^*}^{ij}(\chi_{\kappa,\phi} \circ \kappa_\phi \circ \kappa_{\phi^*}^{-1}; \kappa_{\phi^*} \circ \kappa_\phi^{-1}(\mathbf{X})), \end{aligned} \quad (1.29)$$

and so $\mathcal{F}_{\kappa,\phi}^{ij}(\bullet; \mathbf{X}) \neq \mathcal{F}_{\kappa^*,\phi^*}^{ij}(\bullet; \mathbf{X})$, in contrast to eq. (1.25)³. In the referential description, such relation is only be valid if κ_ϕ is equal to κ_{ϕ^*} , so that the joined mappings $\kappa_{\phi^*} \circ \kappa_\phi^{-1}$ and $\kappa_\phi \circ \kappa_{\phi^*}^{-1}$ are identity. For the typical scenario, in which reference configuration is chosen as the initial configuration, the condition $\kappa_\phi = \kappa_{\phi^*}$ means that both frames coincide at time $t=0$.

The condition of material objectivity can also be written in the referential description. Following a similar procedure as above, eq. (1.26) can be reformulated in terms of $\mathcal{F}_{\kappa,\phi}^{ij}$ instead of $\mathcal{F}_\phi^{ij}$:

$$\mathcal{F}_{\kappa,\phi}^{ij}(\chi_{\kappa,\phi}; \mathbf{X}) = Q_k^i(t) Q_l^j(t) \mathcal{F}_{\kappa,\phi}^{kl}(\chi_{\kappa^*,\phi^*} \circ \kappa_{\phi^*} \circ \kappa_\phi^{-1}; \mathbf{X}). \quad (1.30)$$

³Compare eq. (1.29) with eq. (25) from Liu & Sampaio (2014). Clearly, q^* in their eq. (25) was supposed to be $\kappa_j \circ \kappa_i^{-1}$ and not $j \circ i^{-1}$. Corrected version of their original article can now be downloaded from the author's website (Liu, personal communication). We note that in view of this correction the two generally inappropriate assumptions discussed in Remark 9 of Liu & Sampaio (2014) no longer seem to cancel each other out by a "lucky coincidence": the correctness of both assumptions depends on whether or not $\kappa_j \circ \kappa_i^{-1}$ is identity.

Above we mentioned that the condition of material objectivity can be viewed as the equality of stress in a body experiencing two different motions, by interpreting the mappings χ_ϕ and χ_{ϕ^*} as two different motions measured in one frame ϕ (the invariance of stress under superimposed rigid body motions). By inspection of eq. (1.30) we see that in referential description this interpretation is also possible, but $\kappa_{\phi^*} \circ \kappa_\phi^{-1}$ must be identity for such interpretation to be straightforward. In other words, both such motions $\chi_{\kappa, \phi}$ and χ_{κ^*, ϕ^*} must have the same initial configuration.

In the following sections we will need transformation properties of other kinematic quantities than just the velocity and the acceleration. In the theory of solid materials, fundamental role is played by the deformation gradient:

$$\{\mathbb{F}_\phi\}_j^i := \frac{\partial \chi_{\kappa, \phi}^i(\mathbf{X}, t)}{\partial X^j}; \quad \{\mathbb{F}_{\phi^*}\}_j^i := \frac{\partial \chi_{\kappa^*, \phi^*}^i(\mathbf{X}^*, t)}{\partial X_*^j}. \quad (1.31)$$

The change of frame $\mathbf{x} = \mathbf{x}(\mathbf{x}^*, t)$ provides a possibility to express the motion $\mathbf{x} = \chi_{\kappa, \phi}(\mathbf{X}, t)$ in terms of its counterpart $\mathbf{x}^* = \chi_{\kappa^*, \phi^*}(\mathbf{X}^*, t)$, resulting in the identity $\chi_{\kappa, \phi}(\mathbf{X}, t) = \mathbf{x}(\chi_{\kappa^*, \phi^*}(\mathbf{X}^*(\mathbf{X}), t), t)$, where $\mathbf{X}^*(\mathbf{X})$ is the mapping $\kappa_{\phi^*} \circ \kappa_\phi^{-1}$. By differentiating this relation we get the transformation properties of the deformation gradient

$$\{\mathbb{F}_\phi\}_j^i = \frac{\partial x^i(\chi_{\kappa^*, \phi^*}(\mathbf{X}^*(\mathbf{X}), t), t)}{\partial X^j} = \frac{\partial x^i(\mathbf{x}^*, t)}{\partial x_*^k} \frac{\partial \chi_{\kappa^*, \phi^*}^k(\mathbf{X}^*, t)}{\partial X_*^l} \frac{\partial X_*^l(\mathbf{X})}{\partial X^j}. \quad (1.32)$$

Eq. (1.32) simplifies to the traditional $\{\mathbb{F}_\phi\}_j^i = Q^i_k(t) \{\mathbb{F}_{\phi^*}\}_j^k$ when $\kappa_{\phi^*} \circ \kappa_\phi^{-1}$ is identity. For the inverse of the deformation gradient we analogously obtain $\{\mathbb{F}_\phi^{-1}\}_j^i = O^i_k \{\mathbb{F}_{\phi^*}^{-1}\}_l^k \{Q^{-1}(t)\}_j^l$, with the shifter tensor $O_j^i := \partial X^i / \partial X_*^j$ describing the change of frame for the reference configuration. Typically $O_j^i = Q_j^i(0)$.

In the theory of fluids, it is the velocity gradient that is of primary interest. Thanks to the identity

$$\left\{ \frac{\mathcal{D}\mathbb{F}_\phi}{\mathcal{D}t} \mathbb{F}_\phi^{-1} \right\}_j^i = \frac{\partial^2 \chi_{\kappa, \phi}^i(\mathbf{X}, t)}{\partial X^k \partial t} \frac{\partial \{\chi_{\kappa, \phi}^{-1}\}^k(\mathbf{x}, t)}{\partial x^j} = \frac{\partial v_\phi^i(\mathbf{X}, t)}{\partial X^k} \frac{\partial X^k(\mathbf{x}, t)}{\partial x^j} = \{\nabla \vec{v}_\phi\}_j^i, \quad (1.33)$$

with ∇ standing for the gradient with respect to $\mathbf{x}$, we can use the transformation properties of the deformation gradient to infer the transformation properties of

the velocity gradient:

$$\begin{aligned}
\{\nabla \vec{v}_\phi\}_j^i &= \left\{ \frac{\mathcal{D}\mathbb{F}_\phi}{\mathcal{D}t} \mathbb{F}_\phi^{-1} \right\}_j^i = \frac{\mathcal{D} (Q_m^i(t) \{\mathbb{F}_{\phi^*}\}_n^m \{O^{-1}\}_k^n)}{\mathcal{D}t} O_u^k \{\mathbb{F}_{\phi^*}^{-1}\}_v^u \{Q^{-1}(t)\}_j^v \\
&= Q_m^i(t) \frac{\mathcal{D}\{\mathbb{F}_{\phi^*}\}_n^m}{\mathcal{D}t} \{\mathbb{F}_{\phi^*}^{-1}\}_v^n \{Q^{-1}(t)\}_j^v + \frac{dQ_m^i(t)}{dt} \{Q^{-1}(t)\}_j^m \\
&= Q_m^i(t) \{\nabla \vec{v}_{\phi^*}\}_v^m \{Q^{-1}(t)\}_j^v + \Omega_j^i(t),
\end{aligned} \tag{1.34}$$

where O_j^i is no longer present as it was canceled with its inverse in both terms on the second line. Eq. (1.34) resembles the transformation properties of gradients of objective vector fields (cf. eq. (1.10)). However, since the velocity vector is not objective, there is an additional term $\Omega_j^i(t)$ in eq. (1.34), making $\nabla \vec{v}_\phi$ not objective either. The tensor $\Omega_j^i(t)$ describes the relative rotation of frames ϕ and ϕ^* .

We have defined Euclidean change of frame to be any transformation $\mathbf{x}^* = \mathbf{x}^*(\mathbf{x}, t)$ for which Ω_j^i is spatially homogeneous and antisymmetric (see also Appendix 1.A). It is thus a direct consequence of eq. (1.34) that the symmetric part of the velocity gradient, $\mathbb{D}_\phi := \frac{1}{2}[\nabla \vec{v}_\phi + (\nabla \vec{v}_\phi)^T]$, is Euclidean objective, while the complementary antisymmetric part, $\mathbb{W}_\phi := \frac{1}{2}[\nabla \vec{v}_\phi - (\nabla \vec{v}_\phi)^T]$, is not. The symbol T is the transposition⁴. The strain-rate $\mathbb{D}_\phi$ describes the stretching of a given volume element, while the spin $\mathbb{W}_\phi$ describes the rigid body rotation of that volume element. Indeed, the measure of deformation rate $\mathbb{D}_\phi$ is independent of observer, while the observed local rigid rotation $\mathbb{W}_\phi$ differs by the relative rotation of both frames Ω_j^i from observer to observer.

1.2 Objective tensor rates

In the beginning of this chapter, the 1-D analogue of Maxwell viscoelastic fluid was introduced, leading to the equation $S = \eta \dot{\Delta} - (\eta/G) \dot{S}$. Based on this equation it was suggested that the corresponding 3-D constitutive law should be of the form

$$\tau_\phi^{ij}(\chi_{\kappa, \phi}(\mathbf{X}, t), t) = \mathcal{F}_{\kappa, \phi}^{ij}(\chi_{\kappa, \phi}; \mathbf{X}) = 2\eta_\phi \mathbb{D}_\phi^{ij} - \frac{\eta_\phi}{G_\phi} \frac{\mathcal{D}_?(\tau_\phi^{ij})}{\mathcal{D}t}, \tag{1.35}$$

but it was unclear what stress rate to employ, since the material derivative of second-order tensors was argued to be frame dependent. In the first half of this section we use the condition of material objectivity to impose restrictions on the

⁴Note that the gradient of a vector field is a right covariant mixed tensor, while the transpose of it is a left covariant mixed tensor (covariant in its first index and contravariant in the second index), which implies that both tensors generally cannot be added together, as they are of different kind. Let us, for now, continue in the scenario in which ϕ is a Cartesian frame, where we can disregard such subtleties.

possible stress rates. While such procedure eliminates non-objective tensor rates from considerations, it does not provide a hint as to which tensor rate to actually use. In the second half of this section we use the work of Oldroyd (1950) and geometrical considerations to suggest which objective rate results in a material model that fits the traditional derivation of Maxwell body.

First, let us investigate the transformation properties of $2\eta_\phi \mathbb{D}_\phi^{ij}$. The viscosity η_ϕ is not a kinematic quantity, and its transformation properties thus cannot be derived. Instead, η_ϕ must be postulated to be objective scalar, $\eta_\phi = \eta_{\phi^*}$. The product $2\eta_\phi \mathbb{D}_\phi^{ij}$ is then automatically frame indifferent, because strain-rate $\mathbb{D}$ was shown to be objective in the previous section. Note that when only frame indifferent quantities are used in a constitutive law the condition of material objectivity (1.26), resp. (1.30), is automatically satisfied.

Similarly as for the viscosity, the relaxation time $t_{\text{rel}} := \eta_\phi / G_\phi$ must be postulated to be objective scalar, leaving the transformation properties of the second term in eq. (1.35) to be determined by the properties of the proposed stress rate. In the introduction to this chapter, we have argued by qualitative means that material derivative of second-order tensor fields is not objective. By applying $\mathcal{D}/\mathcal{D}t$ to eq. (1.6) we directly obtain its transformation properties:

$$\begin{aligned} \frac{\mathcal{D}a_\phi^{ij}(\mathbf{x}, t)}{\mathcal{D}t} &= Q_k^i(t) Q_l^j(t) \frac{\mathcal{D}a_{\phi^*}^{kl}(\mathbf{x}^*, t)}{\mathcal{D}t} + \frac{dQ_k^i(t)}{dt} Q_l^j(t) a_{\phi^*}^{kl}(\mathbf{x}^*, t) + Q_k^i(t) \frac{dQ_l^j(t)}{dt} a_{\phi^*}^{kl}(\mathbf{x}^*, t) \\ &= Q_k^i(t) Q_l^j(t) \frac{\mathcal{D}a_{\phi^*}^{kl}(\mathbf{x}^*, t)}{\mathcal{D}t} + \Omega_k^i(t) a_\phi^{kj}(\mathbf{x}, t) + \Omega_k^j(t) a_\phi^{ik}(\mathbf{x}, t), \end{aligned} \quad (1.36)$$

where the objectivity of $\vec{a}$ and the definition $\Omega_k^i := (dQ_k^i/dt) \{Q^{-1}\}_j^k$ are used on the second line. Eq. (1.36) shows that $\mathcal{D}\vec{a}_\phi/\mathcal{D}t$ is not an objective tensor⁵, because its components do not transform according to eq. (1.6). To fix this, one needs to incorporate a quantity into the definition of the proposed time derivative, whose transformation properties could cancel out the second and third terms in eq. (1.36). Eq. (1.34) immediately suggests such quantity. Adding $-\{\nabla v_\phi\}_k^i a_\phi^{kj}$ into the definition of a tensor rate would cancel the second term and adding $-a_\phi^{ik} \{\nabla(v_\phi)\}_k^j$ would cancel the third term from eq. (1.36).

This idea can be formalized by introducing so-called “upper convected”, or Oldroyd, time derivative

$$\frac{\mathcal{D}_{\text{uc}} a_\phi^{ij}}{\mathcal{D}t} := \frac{\mathcal{D}a_\phi^{ij}}{\mathcal{D}t} - \{\nabla \vec{v}_\phi\}_k^i a_\phi^{kj} - a_\phi^{ik} \{\nabla \vec{v}_\phi\}_k^j, \quad (1.37)$$

where a_ϕ^{ij} are the components of any contravariant tensor $\vec{a}$. By combining

⁵To be fully consistent, one should thus distinguish between $\mathcal{D}_\phi(\bullet)/\mathcal{D}t$ and $\mathcal{D}_{\phi^*}(\bullet)/\mathcal{D}t$ in the definition of material time derivative. For brevity we not do so. The observer performing the derivative will always be identifiable from the subscript of the quantity being differentiated.

eqs. (1.34) and (1.36) it follows that

$$\frac{\mathcal{D}_{\text{uc}} a_{\phi}^{ij}(\mathbf{x}, t)}{\mathcal{D}t} = Q_k^i(t) Q_l^j(t) \frac{\mathcal{D}_{\text{uc}} a_{\phi^*}^{kl}(\mathbf{x}^*, t)}{\mathcal{D}t}, \quad (1.38)$$

which means that the Oldroyd derivative of an objective tensor is an objective tensor (the rate is objective). Recall that the transformation properties (1.38) simply imply that

$$\frac{\mathcal{D}_{\text{uc}} \vec{a}_{\phi}(\mathbf{x}, t)}{\mathcal{D}t} = \frac{\mathcal{D}_{\text{uc}} \vec{a}_{\phi^*}(\mathbf{x}^*, t)}{\mathcal{D}t}. \quad (1.39)$$

An immediate consequence is that when such rate is used in eq. (1.35), the resulting constitutive law satisfies the condition of material objectivity.

In Section 1.1 the symmetric and antisymmetric parts of the velocity gradient were introduced and called the strain-rate $\mathbb{D}_{\phi}$ and the spin $\mathbb{W}_{\phi}$ respectively. It was argued that the carrier of non-objectivity of the velocity gradient is the spin $\mathbb{W}_{\phi}$ (because Ω_j^i in eq. (1.34) is antisymmetric). In effect, it is not necessary to use the full velocity gradient when defining a stress rate to get rid of the second and third terms from eq. (1.36). The antisymmetric part $\mathbb{W}_{\phi}$ alone can be employed. This idea stands behind the definition of another rate, named after Jaumann:

$$\frac{\mathcal{D}_{\text{Jau}} a_{\phi}^{ij}}{\mathcal{D}t} := \frac{\mathcal{D} a_{\phi}^{ij}}{\mathcal{D}t} - \mathbb{W}_{\phi}^{ik} a_{\phi}^{kj} + a_{\phi}^{ik} \mathbb{W}_{\phi}^{kj}, \quad (1.40)$$

where the antisymmetry of spin was used, $\mathbb{W}_{\phi}^{jk} = -\mathbb{W}_{\phi}^{kj}$. Again, by combining eqs. (1.34) and (1.36), together with the definition of spin, it can be shown that the Jaumann rate is objective.

In fact, an entire class of objective rates can be built, which the above rates belong to:

$$\frac{\mathcal{D}_{\text{Old}(c)} a_{\phi}^{ij}}{\mathcal{D}t} := \frac{\mathcal{D} a_{\phi}^{ij}}{\mathcal{D}t} - \mathbb{W}_{\phi}^{ik} a_{\phi}^{kj} + a_{\phi}^{ik} \mathbb{W}_{\phi}^{kj} - c (\mathbb{D}_{\phi}^{ik} a_{\phi}^{kl} + a_{\phi}^{ik} \mathbb{D}_{\phi}^{kl}), \quad (1.41)$$

where the parameter c is usually taken to be from the interval $[-1, 1]$. But it can be arbitrary, because both $\vec{a}$ and $\mathbb{D}$ are objective, so their products do not influence the transformation properties of $\mathcal{D}_{\text{Old}(c)}/\mathcal{D}t$. For $c=1$ eq. (1.41) gives the upper convected rate, for $c=0$ we get the Jaumann rate, and for $c=-1$ we will refer to the rate as the lower convected rate and denote it $\mathcal{D}_{\text{lc}}/\mathcal{D}t$.

A different approach that leads to formulating an objective stress rate, namely the Truesdell rate, is to perform material derivative to the second Piola-Kirchhoff stress instead of the Cauchy stress tensor. The two stresses are related through so-called Piola transformation, which is a special mapping of tensors from current to reference configuration of the body. By altering this procedure, in particular by modifying the Piola transformation in a way as to consider only the rotation tensor instead of the full deformation gradient, another objective rate is derived, named the Green-Naghdi rate. We will not discuss these two rates here, with

the exception of one remark in Appendix 1.B. Nevertheless, it is worth noting that the Truesdell rate is equal to the upper convected rate for isochoric flows and that the Green-Naghdi rate resembles the Jaumann rate, only it employs a slightly different kinematic quantity than the spin.

A fundamental insight into objective tensor rates is provided in the work of Oldroyd (1950), who analyzes the temporal evolution of second-order tensors under a very special non-Euclidean change of frame: he considers a frame that is convected with the material. In other words, the coordinate lines of such frame are “frozen” into the material and deform the same way as the body deforms. It is shown that the components of a contravariant tensor $\overleftarrow{a}$ are time constant in the convected frame if and only if the Oldroyd derivative of the tensor is zero:

$$\frac{\mathcal{D}_{uc} \overleftarrow{a}_\phi}{\mathcal{D}t} = 0 \quad \Rightarrow \quad a_{\phi^*}^{ij}(\mathbf{x}^*, t) = a_{\phi^*}^{ij}(\mathbf{x}^*), \quad (1.42)$$

where ϕ^* denotes the convected frame. Note that the properties of the convected frame are governed by the deformation of the body and cannot be arbitrarily chosen. It is generally a curvilinear frame with spatially heterogeneous and temporally dependent metric tensor (see below).

We do not repeat the analysis of Oldroyd here (for a modern treatment, see Gurtin et al., 2010, sections 13 and 20). Instead, the validity of (1.42) is demonstrated in Appendix 1.B on a particular example of stationary simple shearing. Eq. (1.42) is usually interpreted in the way that the natural rate of contravariant tensors is the upper convected rate. It is because the basic property one would expect from a time derivative is that when it is zero the differentiated quantity stays constant. For geometrical interpretation of the constancy of $a_{\phi^*}^{ij}(t)$ see Appendix 1.B below eq. (1.93). The natural rate of covariant tensors is the lower convected rate.

The key question to ask is whether or not there is a particular objective rate that should be used in the constitutive law for Maxwell viscoelastic fluid. Before we address this question, it is necessary to mention “raising and lowering indices” and direct notation.

For any one-form $\underline{f} := f_i \underline{e}^i$ the Riesz representation theorem uniquely gives vector $\vec{f}$ such that $\underline{f}(\vec{v}) = \vec{f} \cdot \vec{v}$ for any $\vec{v} \in \mathbb{V}$, where dot is the scalar product. No particular frame is now under consideration, so the notion of frame is omitted for this paragraph. The components of such vector $\vec{f}$ can be obtained as $f^m = g^{mn} f_n$, where g^{mn} is the inverse of the metric tensor $g_{mn} := \vec{e}_m \cdot \vec{e}_n$ (the inverse defined through $g^{mn} g_{nl} = \delta_l^m$). Thanks to this bijection $\mathcal{L}(\mathbb{V}) \leftrightarrow \mathbb{V}$ one can go from contravariant to covariant tensors and vice versa, and one traditionally distinguishes the components of contravariant, covariant and mixed tensors by upper and lower indices in tensor analysis. In fact, it is common to denote the tensors $\mathbb{A}^{ij} \vec{e}_i \otimes \vec{e}_j$, $\mathbb{A}_j^i \vec{e}_i \otimes \underline{e}^j$ and $\mathbb{A}_{ij} \underline{e}^i \otimes \underline{e}^j$, whose components are related via

$\mathbb{A}^i_j = g_{mj}\mathbb{A}^{im}$ and $\mathbb{A}_{ij} = g_{mj}g_{ni}\mathbb{A}^{nm}$, simply as one tensor $\mathbb{A}$ (direct notation)⁶. In direct notation, the product of two tensors $\mathbb{A}$ and $\mathbb{B}$ is denoted as $\mathbb{A}\mathbb{B}$, and it is defined for tensors of any kind. To illustrate, for contravariant components of the product it holds

$$\{\mathbb{A}\mathbb{B}\}^{ij} = \mathbb{A}^i_n \mathbb{B}^{nj} = \mathbb{A}^{in} \mathbb{B}^j_n = g^{kn} \mathbb{A}^i_n \mathbb{B}^j_k = g_{kn} \mathbb{A}^{in} \mathbb{B}^{kj}, \quad (1.43)$$

because $\vec{e}_m \cdot \vec{e}_n = g_{mn}$, while $\underline{e}^i(\vec{e}_j) = \delta^i_j$, and so if upper and lower indices meet Kronecker delta comes out, while components of the metric tensor come out in other cases. To illustrate the validity of eq. (1.43), consider

$$\begin{aligned} (\mathbb{A}^{in} \vec{e}_i \otimes \vec{e}_n) \cdot (\mathbb{B}^{kj} \vec{e}_k \otimes \vec{e}_j) &= \mathbb{A}^{in} \mathbb{B}^{kj} (\vec{e}_n \cdot \vec{e}_k) \vec{e}_i \otimes \vec{e}_j = \mathbb{A}^{in} \mathbb{B}^{kj} g_{kn} \vec{e}_i \otimes \vec{e}_j \\ &= \mathbb{A}^i_n \mathbb{B}^{nj} \vec{e}_i \otimes \vec{e}_j = \mathbb{A}^i_n \mathbb{B}^{kj} (\vec{e}_i \otimes \underline{e}^n) \cdot (\vec{e}_k \otimes \underline{e}^j) = (\mathbb{A}^i_n \vec{e}_i \otimes \underline{e}^n) \cdot (\mathbb{B}^{kj} \vec{e}_k \otimes \underline{e}^j). \end{aligned}$$

The objective rates discussed above can also be written in direct notation. With the velocity gradient denoted by $\mathbb{L}_\phi$, implication (1.42) reads in direct notation as

$$\frac{\mathcal{D}\mathbb{A}}{\mathcal{D}t} - \mathbb{L}_\phi \mathbb{A} - \mathbb{A} \mathbb{L}_\phi^T = 0 \quad \Rightarrow \quad \mathbb{A}_{\phi^*}^{ij}(\mathbf{x}^*, t) = \mathbb{A}_{\phi^*}^{ij}(\mathbf{x}^*), \quad (1.44)$$

where $\mathbb{A}$ is any objective tensor, and similarly it holds for the lower convected derivative that

$$\frac{\mathcal{D}\mathbb{A}}{\mathcal{D}t} + \mathbb{L}_\phi^T \mathbb{A} + \mathbb{A} \mathbb{L}_\phi = 0 \quad \Rightarrow \quad \{\mathbb{A}_{\phi^*}\}_{ij}(\mathbf{x}^*, t) = \{\mathbb{A}_{\phi^*}\}_{ij}(\mathbf{x}^*). \quad (1.45)$$

Frame ϕ^* is the convected frame in both theorems. In eqs (1.44) and (1.45) the definitions of objective rates are broadened with respect to (1.37) and (1.41), because the rates are now defined for tensors of any kind. The direct notation, however, removes certain subtleties.

Geometrically, tensors are defined as linear mappings from and to $\mathbb{V}$ and $\mathcal{L}(\mathbb{V})$, depending on their kind. The gradient of a vector field is a mapping $\mathbb{V} \rightarrow \mathbb{V}$, its transpose is thus a mapping $\mathcal{L}(\mathbb{V}) \rightarrow \mathcal{L}(\mathbb{V})$. Since $\mathbb{L}_\phi$ is the gradient of the velocity field, only for contravariant tensors $\vec{a}: \mathcal{L}(\mathbb{V}) \rightarrow \mathbb{V}$ it is true that both products $(\{\mathbb{L}_\phi\}_j^i \vec{e}_i \otimes \underline{e}^j) \cdot (a^{kl} \vec{e}_k \otimes \vec{e}_l)$ and $(a^{ij} \vec{e}_i \otimes \vec{e}_j) \cdot (\{\mathbb{L}_\phi\}_l^k \underline{e}^l \otimes \vec{e}_k)$ are again contravariant, that is, the same tensor kind as $\vec{a}$. Similarly, only for covariant tensors $\mathbb{A}_{ij} \underline{e}^i \otimes \underline{e}^j$ both products $(\{\mathbb{L}_\phi\}_l^k \underline{e}^l \otimes \vec{e}_k) \cdot (\mathbb{A}_{ij} \underline{e}^i \otimes \underline{e}^j)$ and $(\mathbb{A}_{kl} \underline{e}^k \otimes \underline{e}^l) \cdot (\{\mathbb{L}_\phi\}_j^i \vec{e}_i \otimes \underline{e}^j)$ are again covariant. This is deeply related to the derivation of theorems (1.44) and (1.45), which are covered in most modern treaties on continuum mechanics (e.g. Gurtin et al. (2010), sections 13 and 20). In analogy to the above, one can directly guess

⁶This may not be interpreted in the way that they are one tensor. Of course, $\mathbb{A}^{ij} \vec{e}_i \otimes \vec{e}_j \neq \mathbb{A}^i_j \vec{e}_i \otimes \underline{e}^j \neq \mathbb{A}_{ij} \underline{e}^i \otimes \underline{e}^j$. Direct notation only uses the fact that a tensor equation for one kind of tensor can be rewritten for any kind of tensor, by raising or lowering indices of tensors appearing in that equation.

the natural rate for right covariant mixed tensors

$$\frac{\mathcal{D}\mathbb{A}}{\mathcal{D}t} - \mathbb{L}_\phi \mathbb{A} + \mathbb{A} \mathbb{L}_\phi = 0 \quad \Rightarrow \quad \{\mathbb{A}_{\phi^*}\}_j^i(\mathbf{x}^*, t) = \{\mathbb{A}_{\phi^*}\}_j^i(\mathbf{x}^*), \quad (1.46)$$

where ϕ^* is the convected frame.

The equation of motion (1.20) is by its nature a vector equation, because acceleration is a vector, and we have thus assumed the Cauchy stress tensor to be contravariant⁷ throughout the text. One could thus argue that it is the upper convected rate which should be used in the definition of Maxwell viscoelastic body. On the other hand, the possibility to change tensor kinds by raising or lowering indices can be used to argue that there is no *a priori* way of determining which objective rate to use, leaving the choice to experiments. It is because both the equation of motion and the constitutive law can be written in direct notation, eradicating the differences between different tensor kinds. In other words, one can write the equation of motion in a vector form, with the contravariant components of the Cauchy stress tensor, and the constitutive law in, for instance, its covariant form, with the covariant components of the Cauchy stress, and combine both equations through raising and lowering indices. This line of thought believes that there is no reason to assume that the constitutive law should be written in terms of the contravariant components of the involved tensors, or in terms of any other particular components for that matter.

We agree with the latter only partially. In theorems (1.44), (1.45) and (1.46) the direct notation can be used only on their left hand sides. It is the constancy of the particular tensor components in the convected frame that the theorems imply, regardless of which components of $\mathbb{A}$ are used when evaluating the objective rates. To illustrate, one could use the covariant components of $\mathbb{A}$ when solving $(\mathcal{D}_{\text{uc}}\mathbb{A})/(\mathcal{D}t)$ in the frame ϕ , and still it would be the contravariant components of $\mathbb{A}$ in the convected frame that would be time constant, and not the covariant components. The main point we are trying to make is that the different objective rates do have different geometrical meaning⁸, and the question to ask is whether there is a particular one, whose geometrical meaning suits the idea of Maxwell viscoelastic fluid outlined in the beginning of this chapter.

Recall from eq. (1.35) that the elastic component of Maxwell viscoelastic rheology must satisfy

$$\frac{\mathcal{D}_?}{\mathcal{D}t} \left(\frac{\tau_\phi^{ij}}{2G} \right) = \mathbb{D}_{\phi, \text{el}}^{ij}. \quad (1.47)$$

⁷Because it must return a vector and acts on normals to surfaces. Normals to surfaces are one-forms, or covectors, because they are the normalized gradients of functions that implicitly define the respective surfaces.

⁸Note that the objective rates from eq. (1.41) have a distinct geometrical meaning only for $c \in \{-1, 0, 1\}$. Using any other value of c in a proposed constitutive law for some material would thus be difficult to justify. One should always consider the geometrical meaning of the objective rate he proposes to employ in a constitutive law.

In linear elastic rheologies, stress is proportional to elastic strain with the constant of proportion equal to $2G$. By replacing $\tau_{\phi}^{ij}/2G$ with $\mathbb{E}_{\phi, \text{el}}^{ij}$ in eq. (1.47) we see that the problem of finding the right objective rate is in fact equivalent to finding a measure of elastic strain $\mathbb{E}_{\phi, \text{el}}$ for which it holds that

$$\frac{\mathcal{D}_? \mathbb{E}_{\phi, \text{el}}^{ij}}{\mathcal{D}t} = \mathbb{D}_{\phi, \text{el}}^{ij}. \quad (1.48)$$

Indeed, the problem in eq. (1.2) was in going from Δ_{elastic} to its rate $\dot{\Delta}_{\text{elastic}}$. One of the most common measures of strain used in continuum mechanics is the Eulerian strain tensor, defined as

$$\mathbb{E}_{\phi} := \frac{1}{2} (\mathbb{I} - \mathbb{F}_{\phi}^{-\text{T}} \mathbb{F}_{\phi}^{-1}). \quad (1.49)$$

It is related to the change of distance between two infinitesimally close material points p and q , whose coordinates differ by $dx^i(t) = \chi_{\phi}^i(p, t) - \chi_{\phi}^i(q, t)$, as follows:

$$\begin{aligned} ds_{\phi}^2 - dS_{\phi}^2 &= g_{ij}^{\phi} dx^i dx^j - g_{ij}^{\phi} dX^i dX^j = g_{ij}^{\phi} dx^i dx^j - g_{ij}^{\phi} \{\mathbb{F}_{\phi}^{-1}\}_k^i \{\mathbb{F}_{\phi}^{-1}\}_l^j dx^k dx^l \\ &= (g_{kl}^{\phi} - g_{ij}^{\phi} \{\mathbb{F}_{\phi}^{-1}\}_k^i \{\mathbb{F}_{\phi}^{-1}\}_l^j) dx^k dx^l = 2\{\mathbb{E}_{\phi}\}_{kl} dx^k dx^l. \end{aligned} \quad (1.50)$$

The Eulerian strain is, geometrically speaking, a covariant tensor, which can be seen both from its definition (1.49) and from eq. (1.50). Eq. (1.50) can be written also in the convected frame ϕ^* . The coordinates of material points do not change with time in the frame ϕ^* , $dx_{*}^i(t) = dX_{*}^i$, but the metric tensor of the frame evolves with time, and so eq. (1.50) takes the form

$$ds^2 - dS^2 = [g_{kl}^{\phi^*}(\mathbf{x}^*, t) - g_{kl}^{\phi^*}(\mathbf{x}^*, 0)] dX_{*}^k dX_{*}^l = 2\{\mathbb{E}_{\phi^*}(\mathbf{x}^*, t)\}_{kl} dX_{*}^k dX_{*}^l \quad (1.51)$$

Eq. (1.51) provides a good reason to use the lower convected rate in Maxwell viscoelastic rheology. When one demands the elastic part of the total deformation to be constant (i.e. the limiting case of purely viscous fluid, $G \rightarrow \infty$), then it is naturally the covariant components of the elastic strain that have to be constant in the convected frame. And this is ensured, for any tensor field, by setting its lower convected derivative to zero.

Coming back to eq. (1.48), we see that making the choice of the lower convected derivative is equivalent to saying that

$$\frac{\mathcal{D}_{\text{lc}} \mathbb{E}_{\phi, \text{el}}}{\mathcal{D}t} = \mathbb{D}_{\phi, \text{el}}. \quad (1.52)$$

Is eq. (1.52) reasonable? It is indeed. For any deformation, the Eulerian strain tensor and the strain-rate tensor are related through the identity (e.g. Martinec, 2011, eq. 2.24):

$$\frac{\mathcal{D} \mathbb{E}_{\phi}}{\mathcal{D}t} = \mathbb{D}_{\phi} - (\mathbb{E}_{\phi} \mathbb{L}_{\phi} + \mathbb{L}_{\phi}^{\text{T}} \mathbb{E}_{\phi}) \quad \Rightarrow \quad \frac{\mathcal{D}_{\text{lc}} \mathbb{E}_{\phi}}{\mathcal{D}t} = \mathbb{D}_{\phi}, \quad (1.53)$$

where the implication follows directly from the definition of the lower convected derivative.

We are aware that the considerations above have certain drawbacks. The major one is that adding strain-rates as $\mathbb{D}_\phi = \mathbb{D}_{\phi,\text{vis}} + \mathbb{D}_{\phi,\text{el}}$ is disputable, which is illustrated in the next section. We do not claim that the lower convected derivative is the only one to use, the goal of our analysis is to discuss the different geometrical meaning of several objective rates in the context of the traditional derivation of Maxwell body. As far as we understand it, the traditional derivation is based on the addition of strain-rates. The identity (1.53) nicely fits into such framework.

In Chapter 3 we use the Jaumann rate in geodynamical modelling. In direct notation the Jaumann rate reads

$$\frac{\mathcal{D}_{\text{Jau}} \mathbb{A}}{\mathcal{D}t} := \frac{\mathcal{D}\mathbb{A}}{\mathcal{D}t} - \mathbb{W}_\phi \mathbb{A} + \mathbb{A} \mathbb{W}_\phi. \quad (1.54)$$

Note that, in view of eq. (1.43), the definition (1.40) is valid only for Cartesian frames. The Jaumann rate is not the natural rate for any tensor kind. Setting this rate to zero ensures that both the covariant and the contravariant components of $\mathbb{A}$ are constant, but not in the convected frame (where this can generally never happen due to the properties of the frame's metric tensor). The components of $\mathbb{A}$ are constant in so-called corotated frame when the Jaumann rate of $\mathbb{A}$ is zero (see the worked-out example in Appendix 1.B). While such property gives the Jaumann rate a clear geometrical interpretation, it is primarily the fact that it preserves the trace of the stress tensor that makes it popular among modelers.

1.3 Maximization of entropy production

The approach used in the preceding sections was based on mechanical considerations. *A posteriori* it can be shown that the Maxwell model derived above satisfies the second law of thermodynamics, that is, entropy is produced when the body deforms. In modern literature, a reverse procedure starts to play increasingly important role in constitutive theory: first it is postulated how a model produces entropy and from that the rheology is derived.

The approach presented in this section is summarized in detail in Málek & Průša (2016). We refer the reader who is not yet familiar with the procedure to their handbook and the references therein. Here we only repeat the analysis of section 4.4.3 in Málek & Průša (2016), where viscoelastic models are derived, and comment on the inclusion of PMFI into the theory.

Similarly as in the traditional approach, the total deformation of the body is decomposed into two parts: dissipative and reversible (elastic). Contrary to the traditional approach, the idea is formalized by introducing so-called “natural configuration” of the body (see Fig. 1.3). The motion mapping the reference

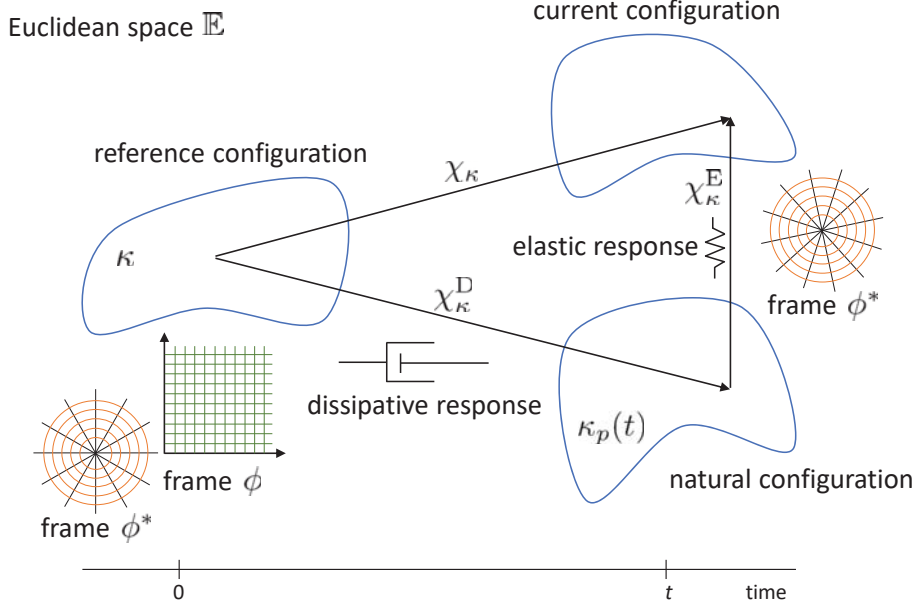


Figure 1.3: Reference, current and natural configurations associated with a viscoelastic body, as viewed from a fixed Cartesian frame ϕ and from a moving polar frame ϕ^* . For a description of the concept of change of frame see also Fig. 1.2.

configuration onto the natural configuration describes the dissipative processes. The mapping from the natural configuration to the current configuration describes the instantaneous elastic response. Every deformation of a body is driven by some loading. Had all the loading suddenly ceased to exist, the body would take the natural configuration.

To formalize this idea, the motion of the body $\chi_{\kappa,\phi}$ is decomposed into two, $\chi_{\kappa,\phi} = \chi_{\kappa_p(t),\phi}^E \circ \chi_{\kappa,\phi}^D$, with $\chi_{\kappa,\phi}^D(\mathbf{X}, t)$ being the dissipative part of the motion, and $\chi_{\kappa_p(t),\phi}^E(\boldsymbol{\xi}, t)$ being the elastic deformation of the natural configuration $\kappa_p(t)$. We denote the coordinates in the natural configuration by $\boldsymbol{\xi}$, $\boldsymbol{\xi}(\mathbf{X}, t) := \chi_{\kappa,\phi}^D(\mathbf{X}, t)$. For the total deformation gradient the chain rule gives

$$\mathbb{F}_\phi = \hat{\mathbb{F}}_\phi \tilde{\mathbb{F}}_\phi, \quad (1.55)$$

where the partial deformation gradients are defined as

$$\{\tilde{\mathbb{F}}_\phi\}_j^i := \frac{\partial \{\chi_{\kappa,\phi}^D(\mathbf{X}, t)\}^i}{\partial X^j}; \quad \{\hat{\mathbb{F}}_\phi\}_j^i := \frac{\partial \{\chi_{\kappa_p(t),\phi}^E(\boldsymbol{\xi}, t)\}^i}{\partial \xi^j}. \quad (1.56)$$

For later reference, we derive a few identities related to kinematics of media with multiple configurations. In analogy with eq. (1.33), the velocity gradient of the natural configuration is defined as

$$\tilde{\mathbb{L}}_\phi := \frac{\mathcal{D}\tilde{\mathbb{F}}_\phi}{\mathcal{D}t} \tilde{\mathbb{F}}_\phi^{-1}. \quad (1.57)$$

Eq. (1.55) together with the identity

$$\frac{\mathcal{D}}{\mathcal{D}t} \left(\tilde{\mathbb{F}}_\phi^{-1} \right) = -\tilde{\mathbb{F}}_\phi^{-1} \frac{\mathcal{D}\tilde{\mathbb{F}}_\phi}{\mathcal{D}t} \tilde{\mathbb{F}}_\phi^{-1} \quad (1.58)$$

yields

$$\frac{\mathcal{D}\hat{\mathbb{F}}_\phi}{\mathcal{D}t} = \frac{\mathcal{D}\mathbb{F}_\phi}{\mathcal{D}t} \tilde{\mathbb{F}}_\phi^{-1} + \mathbb{F}_\phi \frac{\mathcal{D}\tilde{\mathbb{F}}_\phi^{-1}}{\mathcal{D}t} = \mathbb{L}_\phi \mathbb{F}_\phi \tilde{\mathbb{F}}_\phi^{-1} - \mathbb{F}_\phi \tilde{\mathbb{F}}_\phi^{-1} \frac{\mathcal{D}\tilde{\mathbb{F}}_\phi}{\mathcal{D}t} \tilde{\mathbb{F}}_\phi^{-1} = \mathbb{L}_\phi \hat{\mathbb{F}}_\phi - \hat{\mathbb{F}}_\phi \mathbb{L}_\phi. \quad (1.59)$$

For the left Cauchy-Green tensor $\hat{\mathbb{B}}_\phi := \hat{\mathbb{F}}_\phi \hat{\mathbb{F}}_\phi^T$ eq. (1.59) gives

$$\frac{\mathcal{D}\hat{\mathbb{B}}_\phi}{\mathcal{D}t} = \mathbb{L}_\phi \hat{\mathbb{B}}_\phi + \hat{\mathbb{B}}_\phi \mathbb{L}_\phi^T - 2\hat{\mathbb{F}}_\phi \tilde{\mathbb{D}}_\phi \hat{\mathbb{F}}_\phi^T, \quad (1.60)$$

which implies a relation for the material derivative of the trace of $\hat{\mathbb{B}}_\phi$,

$$\frac{\mathcal{D}}{\mathcal{D}t} (\text{Tr} \hat{\mathbb{B}}_\phi) = 2\mathbb{D}_\phi : \hat{\mathbb{B}}_\phi - 2\hat{\mathbb{C}}_\phi : \tilde{\mathbb{D}}_\phi \quad (1.61)$$

where $\mathbb{A}:\mathbb{B} := \text{Tr}(\mathbb{A}\mathbb{B}^T)$ and the identity $\text{Tr}(\mathbb{A}\mathbb{B}) = \text{Tr}(\mathbb{B}\mathbb{A})$ was used to replace the deformation gradients in the second term by the right Cauchy-Green tensor $\hat{\mathbb{C}}_\phi$. Finally, since $\mathcal{D}(\det \mathbb{A})/\mathcal{D}t = (\det \mathbb{A}) \text{Tr}(\mathcal{D}\mathbb{A}/\mathcal{D}t \mathbb{A}^{-1})$, for the material derivative of the logarithm of determinant of the right Cauchy-Green tensor $\hat{\mathbb{B}}$ one gets

$$\frac{\mathcal{D}}{\mathcal{D}t} (\log(\det \hat{\mathbb{B}}_\phi)) = \text{Tr} \left(\frac{\mathcal{D}\hat{\mathbb{B}}_\phi}{\mathcal{D}t} \hat{\mathbb{B}}_\phi^{-1} \right) = 2\mathbb{I} : \mathbb{D} - 2\mathbb{I} : \tilde{\mathbb{D}}_\phi \quad (1.62)$$

Often, the only quantity that needs to be specified in the procedure by (Rajagopal & Srinivasa, 2004) is the production of entropy. For viscoelastic materials, however, the internal energy of the body must be allowed to account for stored elastic energy. The form of such storage is also to be assumed, and here the internal energy takes the following form:

$$e(s, \rho, \text{Tr} \hat{\mathbb{B}}_\phi, \det \hat{\mathbb{B}}_\phi) = \tilde{e}(s, \rho) + \frac{G}{2\rho} \left(\text{Tr} \hat{\mathbb{B}}_\phi - 3 - \log(\det \hat{\mathbb{B}}_\phi) \right). \quad (1.63)$$

The second term on the right hand side represents the energy stored in the elastic deformation of the natural configuration and $\tilde{e}$ is analogous of the internal energy for a simple thermodynamic system, for which it depends on entropy and volume of the system only. The quantity s is the specific entropy per unit volume. The motivation for this particular choice of storage of elastic energy comes from the theory of isotropic elastic materials (see Horgan & Saccomandi, 2004, for a list of other frequently used possibilities).

Next step in the procedure is to derive evolutionary equation for the specific entropy of the model. By applying material derivative to eq. (1.63), and

multiplying both sides by ρ , we get,

$$\begin{aligned}\rho \frac{\mathcal{D}e}{\mathcal{D}t} &= \rho \left(\frac{\partial e}{\partial s} \frac{\mathcal{D}s}{\mathcal{D}t} + \frac{\partial e}{\partial \rho} \frac{\mathcal{D}\rho}{\mathcal{D}t} + \frac{\partial e}{\partial (\text{Tr } \hat{\mathbb{B}}_\phi)} \frac{\mathcal{D}(\text{Tr } \hat{\mathbb{B}}_\phi)}{\mathcal{D}t} + \frac{\partial e}{\partial (\det \hat{\mathbb{B}}_\phi)} \frac{\mathcal{D}(\det \hat{\mathbb{B}}_\phi)}{\mathcal{D}t} \right) \\ &= \rho \frac{\partial \tilde{e}}{\partial s} \frac{\mathcal{D}s}{\mathcal{D}t} + \left(\rho \frac{\partial \tilde{e}}{\partial \rho} - \frac{G}{2\rho} (\text{Tr } \hat{\mathbb{B}}_\phi - 3 - \log(\det \hat{\mathbb{B}}_\phi)) \right) \frac{\mathcal{D}\rho}{\mathcal{D}t} \\ &\quad + G \left(\hat{\mathbb{B}}_\phi : \mathbb{D}_\phi - \hat{\mathbb{C}}_\phi : \tilde{\mathbb{D}}_\phi - \mathbb{D}_\phi : \mathbb{I} + \tilde{\mathbb{D}}_\phi : \mathbb{I} \right), \quad (1.64)\end{aligned}$$

where eqs (1.61) and (1.62) are used on the third line. Along with Málek & Průša (2016), we define the “pressure” as

$$p_{\text{th}}^{\text{M}} := \rho^2 \frac{\partial e}{\partial \rho} = \rho^2 \frac{\partial \tilde{e}}{\partial \rho} - \frac{G}{2} \left(\text{Tr } \hat{\mathbb{B}}_\phi - 3 - \log(\det \hat{\mathbb{B}}_\phi) \right). \quad (1.65)$$

Note that such definition is questionable. Thermodynamic pressure is the system’s resistance to changes of volume, generally defined as in eq. (1.65), but under the assumption that the internal energy e is known as a function of mutually independent state variables. In our case, $\hat{\mathbb{B}}_\phi$ describes the deformation of the natural configuration, and $\det \hat{\mathbb{B}}_\phi$ is thus inevitably related to changes of density of our medium. Since the density ρ and $\det \hat{\mathbb{B}}_\phi$ are not mutually independent state variables, we believe that the p_{th}^{M} , as defined in eq. (1.65), is not the thermodynamic pressure of the studied system. Nevertheless, by employing eq. (1.65), together with the definition of temperature, $\theta := \partial e / \partial s$, and the continuity equation, eq. (1.64) simplifies to

$$\rho \frac{\mathcal{D}e}{\mathcal{D}t} = \rho \theta \frac{\mathcal{D}s}{\mathcal{D}t} - p_{\text{th}}^{\text{M}} \text{div } \vec{v}_\phi + G \left(\hat{\mathbb{B}}_\phi : \mathbb{D}_\phi - \hat{\mathbb{C}}_\phi : \tilde{\mathbb{D}}_\phi - \mathbb{D}_\phi : \mathbb{I} + \tilde{\mathbb{D}}_\phi : \mathbb{I} \right). \quad (1.66)$$

The temporal evolution of the specific internal energy e is, on the other hand, also governed by the standard energy balance

$$\rho \frac{\mathcal{D}e}{\mathcal{D}t} = \boldsymbol{\tau}_\phi : \mathbb{D}_\phi - \text{div } \vec{j}_q, \quad (1.67)$$

where $\vec{j}_q$ is the heat flux. Internal (e.g. radioactive) heat sources are omitted. By inserting for $\rho \mathcal{D}e / \mathcal{D}t$ from eq. (1.67) into eq. (1.66) we get

$$\rho \frac{\mathcal{D}s}{\mathcal{D}t} + \text{div } \frac{\vec{j}_g}{\theta} = \frac{1}{\theta} \left[\boldsymbol{\tau}_\phi : \mathbb{D}_\phi - \frac{\vec{j}_q \cdot \nabla \theta}{\theta} + p_{\text{th}}^{\text{M}} \text{div } \vec{v}_\phi - G \left(\hat{\mathbb{B}}_\phi : \mathbb{D}_\phi - \hat{\mathbb{C}}_\phi : \tilde{\mathbb{D}}_\phi - \mathbb{D}_\phi : \mathbb{I} + \tilde{\mathbb{D}}_\phi : \mathbb{I} \right) \right], \quad (1.68)$$

where the identity $\text{div}(\vec{j}_g / \theta) = (\text{div } \vec{j}_g) / \theta - (\vec{j}_g \cdot \nabla \theta) / \theta^2$ was used in order to obtain the entropy production of the system,

$$\xi := \rho \mathcal{D}s / \mathcal{D}t + \text{div}(\vec{j}_g / \theta), \quad (1.69)$$

on the left-hand side of eq. (1.68). The second law of thermodynamics states

that the entropy production is always positive, because the maximum outflux of entropy from a cooling thermodynamic system is given as $\vec{j}_q/\theta$, integrated over the surface of the system. For quasi-static processes $\rho \mathcal{D}s/\mathcal{D}t = -\text{div}(\vec{j}_q/\theta)$ and the entropy production is zero.

Eq. (1.68) is valid for all materials that store elastic energy in the manner specified by eq. (1.63), regardless of their rheology. Right-hand side of eq. (1.68) is traditionally divided into terms representing mutually independent processes, each term usually expressed as a product of a thermodynamic affinity and a thermodynamic flux. For this reason, we separate the trace of the strain-rate, related to volumetric changes during deformation, from the deviatoric part of the strain-rate, which is related to isochoric shearing. The second law of thermodynamics then takes the form

$$\begin{aligned} \theta \xi = & \left(\boldsymbol{\tau}_\phi^{\text{d}} - G \hat{\mathbb{B}}_\phi^{\text{d}} \right) : \mathbb{D}_\phi^{\text{d}} + \left(m + p_{\text{th}}^{\text{M}} - \frac{G}{3} \text{Tr} \hat{\mathbb{B}}_\phi + G \right) \text{div} \vec{v}_\phi + G \hat{\mathbb{C}}_\phi^{\text{d}} : \tilde{\mathbb{D}}_\phi^{\text{d}} \\ & + G \left(\frac{\text{Tr} \hat{\mathbb{B}}_\phi}{3} - 1 \right) \text{Tr} \tilde{\mathbb{D}}_\phi - \frac{\vec{j}_q \cdot \nabla \theta}{\theta} \geq 0, \quad (1.70) \end{aligned}$$

where superscript d denotes the deviatoric part of a tensor and m is the trace of the Cauchy stress tensor $\boldsymbol{\tau}_\phi$.

One way to propose a material rheology is to derive the formula for its entropy production, as done above, and to demand each term in the formula to be positive. This can be illustrated on the example of materials with no storage of elastic energy. In such a scenario, the first term in eq. (1.70) is $\boldsymbol{\tau}_\phi^{\text{d}} : \mathbb{D}_\phi^{\text{d}}$, which can be made positive by proposing a linear relation $\boldsymbol{\tau}_\phi^{\text{d}} \approx \mathbb{D}_\phi^{\text{d}}$, that is, by proposing Newtonian viscous rheology $\boldsymbol{\tau}_\phi^{\text{d}} = 2\eta \mathbb{D}_\phi^{\text{d}}$, the constant of proportion known as viscosity, $\eta \geq 0$. The resulting entropy production term is clearly positive, because $2\eta \mathbb{D}_\phi^{\text{d}} : \mathbb{D}_\phi^{\text{d}} = 2\eta |\mathbb{D}_\phi^{\text{d}}|^2$. In a way, the method by Rajagopal & Srinivasa (2004) is a generalization of such procedure. In their method, instead of inspecting eq. (1.70) and proposing relations between the thermodynamic affinities and fluxes to make each term positive, the entropy production is directly postulated as a function of the affinities. To obtain Maxwell-type fluid, Málek & Průša (2016) propose the following entropy production terms:

$$\xi(\mathbb{D}_\phi^{\text{d}}, \text{div} \vec{v}_\phi, \tilde{\mathbb{D}}_\phi, \nabla \theta) := \frac{1}{\theta} \left[2\eta |\mathbb{D}_\phi^{\text{d}}|^2 + \frac{2\eta + 3\lambda}{3} (\text{div} \vec{v}_\phi)^2 + 2\eta_1 \tilde{\mathbb{D}}_\phi : (\hat{\mathbb{C}}_\phi \tilde{\mathbb{D}}_\phi) + \kappa \frac{|\nabla \theta|^2}{\theta} \right]. \quad (1.71)$$

The constants 2η , $(2\eta + 3\lambda)/3$, $2\eta_1$, and κ must all be greater than or equal to zero in order to satisfy the second law of thermodynamics in a general case. The motivation for the first two terms in eq. (1.71) comes from the theory of compressible Newtonian viscous fluid, as illustrated in the previous paragraph, and the respective constants are known as the shear and bulk viscosity. The last term in eq. (1.71) is the entropy production due to the transfer of heat, and stems from Fourier law.

As discussed at the end of this section, the third term in eq. (1.71) is motivated, so to say, by the desired outcome of the derivation. In the procedure by Rajagopal, it is the two scalar functions e and ξ that fully determine the material properties. When thermodynamic fluxes are given, the procedure assumes that each material behaves so as to maximize its entropy production for the given fluxes. To obtain the material rheology, one thus has to maximize the entropy production (1.71) with respect to the affinities $\mathbb{D}_\phi^d, \text{div} \vec{v}_\phi, \tilde{\mathbb{D}}_\phi, \nabla \theta$, with eq. (1.70) serving as a constraint for the maximization.

As long as the scalar functions e and ξ are objective scalars, the entropy production maximization results in constitutive laws that automatically satisfy the principle of material frame-indifference. This is a serious advantage when compared to the traditional technique. To ensure that e and ξ are objective scalars, one must investigate how the components of the kinematic quantities appearing in eq. (1.71) transform under a change of frame. To our knowledge, this is not discussed in the relatively recent literature that deals with the entropy maximization procedure within the framework including the natural configuration, that is, within the framework involving the composite deformation $\chi_{\kappa, \phi} = \chi_{\kappa_p(t), \phi}^E \circ \chi_{\kappa, \phi}^D$.

Similarly to eq. (1.32), we can derive the transformation properties of deformation gradient $\hat{\mathbb{F}}_\phi$ under a change of frame, $\phi \rightarrow \phi^*$. Recall that $\hat{\mathbb{F}}_\phi$ describes the elastic deformation of the natural configuration, and so

$$\{\hat{\mathbb{F}}_\phi\}_j^i = \frac{\partial x^i(\chi_{\kappa_p^*(t), \phi^*}^E(\xi^*(\xi), t), t)}{\partial \xi^j} = \frac{\partial x^i(\mathbf{x}^*, t)}{\partial x_*^k} \frac{\partial \{\chi_{\kappa_p^*(t), \phi^*}^E(\xi^*, t)\}^k}{\partial \xi_*^l} \frac{\partial \xi_*^l(\xi, t)}{\partial \xi^j}. \quad (1.72)$$

The important difference from eq. (1.32) is that while the reference configuration is a time independent mapping, the natural configuration evolves with time (similarly as the current configuration, see Fig. 1.3). The transformation $\xi \rightarrow \xi^*$ is thus identical to the transformation $\mathbf{x} \rightarrow \mathbf{x}^*$, making $\hat{\mathbb{F}}_\phi$ an objective contravariant tensor (unlike the common deformation gradient $\mathbb{F}_\phi$, which is a non-standard object).

Note that the traditional formula for the total deformation gradient, $\{\mathbb{F}_\phi\}_j^i = Q_k^i(t) \{\mathbb{F}_{\phi^*}\}_j^k$, is not applicable to the partial deformation gradient $\hat{\mathbb{F}}_\phi$. While one can easily restrict their considerations to changes of frame that satisfy $\mathbf{X}^*(\mathbf{X}) = \mathbf{X}$ when discussing objectivity, it is not possible to demand that $\xi^*(\xi, t) = \xi$ (compare eqs (1.32) and (1.72)). Assuming $\xi^*(\xi, t) = \xi$ means nothing less than to demand that both frames ϕ and ϕ^* are identical at all times, prohibition any discussion of objectivity in effect. Note also that, in view of eq. (1.72), the left Cauchy-Green tensor $\hat{\mathbb{C}}_\phi$ is an objective tensor, and not objective scalar, as is sometimes erroneously stated (perhaps because $\{\mathbb{F}_\phi\}_j^i = Q_k^i(t) \{\mathbb{F}_{\phi^*}\}_j^k$ is erroneously assumed to hold for $\hat{\mathbb{F}}_\phi$).

By repeating the argument above, it can be shown that also $\tilde{\mathbb{D}}_\phi$ is an objective tensor. As a result, the entropy production term $2\eta_1 \tilde{\mathbb{D}}_\phi : (\hat{\mathbb{C}}_\phi \tilde{\mathbb{D}}_\phi)$ is an objective scalar as desired. By a coincidence, the same conclusion is reached when the

components of $\hat{\mathbb{C}}_\phi$ are incorrectly assumed to be objective scalars.

The constrained maximization of (1.71) is performed using the Lagrange multipliers, and we refer the reader to Málek & Průša (2016) for more details. It results in the following set of equations:

$$\boldsymbol{\tau}_\phi^d - G \hat{\mathbb{B}}_\phi^d = 2\eta \mathbb{D}_\phi^d \quad (1.73)$$

$$m + p_{\text{th}}^M - \frac{G}{3} \text{Tr} \hat{\mathbb{B}}_\phi + G = \frac{2\eta + 3\lambda}{3} \text{div} \vec{v}_\phi \quad (1.74)$$

$$G \left(\hat{\mathbb{C}}_\phi - \mathbb{I} \right) = 2\eta_1 \hat{\mathbb{C}}_\phi \tilde{\mathbb{D}}_\phi \quad (1.75)$$

$$\vec{j}_q = -\kappa \nabla \theta \quad (1.76)$$

Upon multiplying eq. (1.75) by $\hat{\mathbb{F}}_\phi^T$ from the right and by $\hat{\mathbb{F}}_\phi^T$ from the left, and recalling the identity (1.60), eq. (1.75) reads

$$\frac{\mathcal{D} \hat{\mathbb{B}}_\phi}{\mathcal{D}t} - \mathbb{L}_\phi \hat{\mathbb{B}}_\phi - \hat{\mathbb{B}}_\phi \mathbb{L}_\phi^T + G \left(\hat{\mathbb{B}}_\phi - \mathbb{I} \right) = 0, \quad (1.77)$$

which allows one to combine eqs (1.73), (1.74) and (1.77) into the constitutive relation for the Cauchy stress tensor:

$$\boldsymbol{\tau}_\phi = -p_{\text{th}}^M + 2\eta \mathbb{D}_\phi + \mathbb{S}_\phi + \lambda (\text{div} \vec{v}_\phi) \mathbb{I} \quad (1.78a)$$

$$\frac{\eta_1}{G} \left(\frac{\mathcal{D} \mathbb{S}_\phi}{\mathcal{D}t} - \mathbb{L}_\phi \mathbb{S}_\phi - \mathbb{S}_\phi \mathbb{L}_\phi^T \right) + \mathbb{S}_\phi = 2\eta_1 \mathbb{D}_\phi, \quad (1.78b)$$

where the extra stress tensor $\mathbb{S}_\phi := G(\hat{\mathbb{B}}_\phi - \mathbb{I})$ was introduced. Málek & Průša (2016) note that eq. (1.78a) can be seen as a *compressible variant of the classical Oldroyd-B model*, developed for viscoelastic incompressible fluids by Oldroyd (1950). In our case, it is interesting to set η and λ to zero, because then eq. (1.78a) reduces to the constitutive relation for a compressible Maxwell-type viscoelastic fluid:

$$\boldsymbol{\tau}_\phi = -p_{\text{th}}^M + \mathbb{S}_\phi \quad (1.79a)$$

$$\frac{\eta_1}{G} \left(\frac{\mathcal{D} \mathbb{S}_\phi}{\mathcal{D}t} - \mathbb{L}_\phi \mathbb{S}_\phi - \mathbb{S}_\phi \mathbb{L}_\phi^T \right) + \mathbb{S}_\phi = 2\eta_1 \mathbb{D}_\phi. \quad (1.79b)$$

The resulting constitutive eq. (1.79a) contains the upper convected rate. Unlike in the traditional technique, which is based on mechanical analogues, the stress rate is a result of the performed procedure here. Seemingly, this makes the geometrical interpretations of various objective rates redundant. However, one must realize that choosing the entropy production maximization over mechanical analogues is nothing but choosing one uncertainty over the other. The upper convected derivative in eq. (1.79a) can be traced back to the entropy production term $2\eta_1 \tilde{\mathbb{D}}_\phi : (\hat{\mathbb{C}}_\phi \tilde{\mathbb{D}}_\phi)$ in eq. (1.71). Finding a valid physical interpretation for this dissipational mechanism may be equally hard as trying to justify the upper

convected rate on geometrical grounds. Nevertheless, the maximization technique is very intriguing and various dissipational mechanisms can be expected to appear in the literature. We hope that this section shed more light on the transformation properties of $\hat{\mathbb{C}}_\phi$, $\hat{\mathbb{B}}_\phi$ and $\tilde{\mathbb{D}}_\phi$, which usually have to be considered when proposing an equation for the entropy production ξ .

1.A Change of frame: moving polar coordinates

We consider a 2-D Euclidean space with two frames, one Cartesian and the other having polar coordinates. They are related via

$$\begin{aligned} x &= r \cos(\varphi + \theta(t)) + c^1(t); & r &= \sqrt{(x - c^1(t))^2 + (y - c^2(t))^2}; \\ y &= r \sin(\varphi + \theta(t)) + c^2(t); & \varphi &= \arctan\left(\frac{y - c^2(t)}{x - c^1(t)}\right), \end{aligned}$$

where, $\vec{c}(t)$ and $\theta(t)$ describe the relative movement of both observers. To recall the notation from the main text, $x=x^1, y=x^2, r=x_*^1, \varphi=x_*^2$. The shifter tensor Q^i_j and its inverse are, by their definition,

$$Q(\mathbf{x}^*, t) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos(\varphi + \theta(t)) & -r \sin(\varphi + \theta(t)) \\ \sin(\varphi + \theta(t)) & r \cos(\varphi + \theta(t)) \end{bmatrix} \quad (1.80)$$

$$Q^{-1}(\mathbf{x}, t) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x - c^1(t)}{\sqrt{(x - c^1(t))^2 + (y - c^2(t))^2}} & \frac{y - c^2(t)}{\sqrt{(x - c^1(t))^2 + (y - c^2(t))^2}} \\ \frac{c^2(t) - y}{(x - c^1(t))^2 + (y - c^2(t))^2} & \frac{x - c^1(t)}{(x - c^1(t))^2 + (y - c^2(t))^2} \end{bmatrix}. \quad (1.81)$$

Both matrices are indeed inverse to each other, which can be seen when the conversion $\mathbf{x} \rightarrow \mathbf{x}^*$ is performed in eq. (1.81),

$$Q^{-1} = \begin{bmatrix} \cos(\varphi + \theta(t)) & \sin(\varphi + \theta(t)) \\ -\frac{1}{r} \sin(\varphi + \theta(t)) & \frac{1}{r} \cos(\varphi + \theta(t)) \end{bmatrix}.$$

For the curvilinear local vector basis we get (cf. eq. (1.5))

$$\begin{aligned} \vec{g}_r(\mathbf{x}^*, t) &= \cos(\varphi + \theta(t)) \vec{e}_x + \sin(\varphi + \theta(t)) \vec{e}_y \\ \vec{g}_\varphi(\mathbf{x}^*, t) &= -r \sin(\varphi + \theta(t)) \vec{e}_x + r \cos(\varphi + \theta(t)) \vec{e}_y. \end{aligned} \quad (1.82)$$

Now, let us consider a motion of a single particle p . In the frame ϕ^* it is described as a time-dependent mapping of the particle p into the coordinate system, $\mathbf{x}^* = \chi_{\phi^*}(p, t)$, so for the two coordinates of the particle's trajectory we have $r=r(p, t)$, and $\varphi=\varphi(p, t)$. For the position vector, upon inserting eq. (1.5)

into eq. (1.13), the following relation holds,

$$x^i(p, t) \vec{e}_i = [Q_j^k(\mathbf{x}^*(p, t), t) s_{\phi^*}^j(\mathbf{x}^*(p, t), t) + c^k(t)] \vec{e}_k, \quad (1.83)$$

which leads to an alternative form of eq. (1.14):

$$v_\phi^i \vec{e}_i = \left(Q_j^k \frac{\mathcal{D} s_{\phi^*}^j}{\mathcal{D} t} + \frac{\partial Q_j^k}{\partial x_*^l} \frac{\mathcal{D} x_*^l}{\mathcal{D} t} s_{\phi^*}^j + \frac{\partial Q_j^k}{\partial t} s_{\phi^*}^j + \frac{dc_\phi^k}{dt} \right) \vec{e}_k. \quad (1.84)$$

Note that in polar coordinates the position vector has radial component only, $\vec{s}_{\phi^*}(\mathbf{x}^*, t) = s_{\phi^*}^1(\mathbf{x}^*, t) \vec{g}_r(\mathbf{x}^*, t)$, and so while $s_{\phi^*}^1$ is equal to r , the second component of the position vector $s_{\phi^*}^2$ is identically zero. We can thus simplify the first two terms in eq. (1.84):

$$v_\phi^i \vec{e}_i = \left(Q_1^k \frac{\mathcal{D} r}{\mathcal{D} t} + \frac{\partial Q_1^k}{\partial \varphi} \frac{\mathcal{D} \varphi}{\mathcal{D} t} s_{\phi^*}^1 + \frac{\partial Q_j^k}{\partial t} s_{\phi^*}^j + \frac{dc_\phi^k}{dt} \right) \vec{e}_k, \quad (1.85)$$

where the independence of Q_1^k on r for both $k=1, 2$ was recognized. If we evaluate the expression and employ the relations (1.82) we get

$$v_\phi^i \vec{e}_i = \frac{\mathcal{D} r}{\mathcal{D} t} \vec{g}_r + \frac{\mathcal{D} \varphi}{\mathcal{D} t} \vec{g}_\varphi + \frac{\partial Q_j^k}{\partial t} s_{\phi^*}^j \vec{e}_k + \frac{dc_\phi^k}{dt} \vec{e}_k. \quad (1.86)$$

We see that the first sum in eq. (1.14) from the main text, $(\mathcal{D} s_{\phi^*}^j / \mathcal{D} t) \vec{g}_j$, is equal to $(\mathcal{D} r / \mathcal{D} t) \vec{g}_r$ for our particular example, which is the radial velocity of the particle. The second sum in eq. (1.14), $s_{\phi^*}^j (\partial \vec{g}_j / \partial x_*^l) (\mathcal{D} x_*^l / \mathcal{D} t)$, is equal to $(\mathcal{D} \varphi / \mathcal{D} t) \vec{g}_\varphi$ for our example, which is the angular velocity of the particle. The terms are thus the two components of the velocity vector $\vec{v}_{\phi^*}$. Indeed, they describe the tangent to the parametrized curve $r=r(p, t)$; $\varphi=\varphi(p, t)$, that is, the particle's trajectory, which is the primary definition of velocity⁹.

To investigate the third sum in eq. (1.14), $s_{\phi^*}^j \frac{\partial \vec{g}_j}{\partial t}$, let us insert in eq. (1.86) for $s_{\phi^*}^j$ from eq. (1.83):

$$v_\phi^i \vec{e}_i = \frac{\mathcal{D} r}{\mathcal{D} t} \vec{g}_r + \frac{\mathcal{D} \varphi}{\mathcal{D} t} \vec{g}_\varphi + \frac{\partial Q_{kj}}{\partial t} Q_{jl}^{-1} (x^l - c^l) \vec{e}_k + \frac{dc_\phi^k}{dt} \vec{e}_k. \quad (1.87)$$

⁹Note that $\vec{v}_{\phi^*} \neq \frac{\mathcal{D}}{\mathcal{D} t} (s_{\phi^*}^j(\mathbf{x}^*, t) \vec{g}_j(\mathbf{x}^*, t))$. It is because any observer, when provided the components of the position vector (for each time t) with the task to measure the particle velocity, is generally not aware of the fact that he himself is moving (he measures the velocity with respect to his frame), and so the term $s_{\phi^*}^j(\mathbf{x}^*, t) \partial \vec{g}_j(\mathbf{x}^*, t) / \partial t$, discussed below, is not part of the velocity measured by the observer.

Since the product $\Omega_l^k := (\partial Q^k / \partial t) \{Q^{-1}\}_l^j$ is equal to

$$\begin{bmatrix} -\frac{d\theta}{dt} \sin(\varphi+\theta) & -\frac{d\theta}{dt} r \cos(\varphi+\theta) \\ \frac{d\theta}{dt} \cos(\varphi+\theta) & -\frac{d\theta}{dt} r \sin(\varphi+\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi+\theta) & \sin(\varphi+\theta) \\ -\frac{1}{r} \sin(\varphi+\theta) & \frac{1}{r} \cos(\varphi+\theta) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d\theta}{dt} \\ \frac{d\theta}{dt} & 0 \end{bmatrix},$$

we see that the change of frame studied in this example satisfies our definition of Euclidean change of frame, because Ω_l^k is antisymmetric and spatially homogeneous. Indeed, for any antisymmetric and homogeneous tensor Ω_l^k , the third term in eq. (1.87), resp. in eq. (1.14), can be written as $\xi_{ijk} \omega^j (x^k - c^k) \vec{e}_i$, which is the cross product $\vec{\omega} \times (\vec{s}_\phi - \vec{c}) = \vec{\omega} \times \vec{s}_{\phi^*}$. This term describes the relative rotation of both frames. In other words, the mutual motion of two frames that are bound by Euclidean change of frame is a combination of translation and rotation only, it is a deformationless motion. This can be confirmed by observing the metric tensor of the frame ϕ^* , $g_{mn} := \vec{g}_m \cdot \vec{g}_n$. The metric tensor is spatially dependent, but constant in time:

$$g_{mn}(\mathbf{x}^*, t) = \begin{bmatrix} \vec{g}_r \cdot \vec{g}_r & \vec{g}_r \cdot \vec{g}_\phi \\ \vec{g}_\phi \cdot \vec{g}_r & \vec{g}_\phi \cdot \vec{g}_\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}. \quad (1.88)$$

For the convected frame, discussed below, we cannot write $g_{mn}(\mathbf{x}^*, t) = g_{mn}(\mathbf{x}^*)$, because switching to the convected frame is not a Euclidean change of frame.

1.B Convected and corotated frames: simple shearing

Let us assume a 2-D Euclidean space with a Cartesian frame ϕ , in which simple shearing of a body is observed:

$$\left. \begin{aligned} \chi_{\kappa, \phi}^1(\mathbf{X}, t) &= X^1 + \alpha X_2 t \\ \chi_{\kappa, \phi}^2(\mathbf{X}, t) &= X^2 \end{aligned} \right\} \quad \left. \begin{aligned} v_\phi^1(\mathbf{x}, t) &= \alpha x_2 \\ v_\phi^2(\mathbf{x}, t) &= 0 \end{aligned} \right\} \quad \nabla \vec{v}_\phi = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

Note that $\chi_{\kappa, \phi}^1(\mathbf{X}, t=0) = X^1$ and $\chi_{\kappa, \phi}^2(\mathbf{X}, t=0) = X^2$, and so the reference configuration is the initial configuration. The strain-rate and the spin of this motion are, along with the velocity gradient, spatially homogeneous and stationary:

$$\mathbb{D}_\phi = \frac{1}{2} (\nabla \vec{v}_\phi + (\nabla \vec{v}_\phi)^T) = \begin{pmatrix} 0 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 0 \end{pmatrix}; \quad \mathbb{W}_\phi = \frac{1}{2} (\nabla \vec{v}_\phi - (\nabla \vec{v}_\phi)^T) = \begin{pmatrix} 0 & \frac{\alpha}{2} \\ -\frac{\alpha}{2} & 0 \end{pmatrix}.$$

Let us further assume a contravariant tensor $\vec{a}(\mathbf{x}, t) = a_\phi^{ij}(\mathbf{x}, t) \vec{e}_i \otimes \vec{e}_j$, whose time evolution is governed by

$$\frac{\mathcal{D}_{\text{uc}} a_\phi^{ij}}{\mathcal{D}t} = \frac{\mathcal{D} a_\phi^{ij}}{\mathcal{D}t} - \{\nabla \vec{v}_\phi\}_k^i a_\phi^{kj} - a_\phi^{ik} \{\nabla \vec{v}_\phi\}_k^j = 0. \quad (1.89)$$

At the origin of the frame ($\mathbf{x}=0$), the observed velocity of the motion is zero and eq. (1.89) reduces to

$$\frac{\partial}{\partial t} \begin{bmatrix} a_{\phi}^{11} & a_{\phi}^{12} \\ a_{\phi}^{12} & a_{\phi}^{22} \end{bmatrix} = \begin{bmatrix} 2\alpha a_{\phi}^{12} & \alpha a_{\phi}^{22} \\ \alpha a_{\phi}^{22} & 0 \end{bmatrix}$$

This set of ordinary differential equations has the following solution

$$\begin{aligned} \vec{a}(\mathbf{x}=0, t) = & (2\alpha^2 t^2 a_0^{22} + 2\alpha t a_0^{12} + a_0^{11}) \vec{e}_1 \otimes \vec{e}_1 + (\alpha t a_0^{22} + a_0^{12}) \vec{e}_1 \otimes \vec{e}_2 \\ & + (\alpha t a_0^{22} + a_0^{12}) \vec{e}_2 \otimes \vec{e}_1 + a_0^{22} \vec{e}_2 \otimes \vec{e}_2, \end{aligned} \quad (1.90)$$

where $a_0^{11}, a_0^{22}, a_0^{12}$ are the initial values of $\vec{a}$, that is, $a_{\phi}^{ij}(\mathbf{x}=0, t=0)$.

The simple shearing studied here can be also viewed from the convected frame ϕ^* , related to ϕ through a non-Euclidean change of frame $\mathbf{x}^* = \mathbf{x}^*(\mathbf{x}, t)$, defined so that the coordinates of material points stay constant throughout the motion:

$$\left. \begin{aligned} x^1(\mathbf{x}^*, t) &= x_*^1 + \alpha x_*^2 t \\ x^2(\mathbf{x}^*, t) &= x_*^2 \end{aligned} \right\} \quad \begin{aligned} x_*^1(\mathbf{x}, t) &= x^1 - \alpha x^2 t \\ x_*^2(\mathbf{x}, t) &= x^2 \end{aligned}$$

Thus, in the convected frame the motion reads $\chi_{\kappa^*, \phi^*}^i(\mathbf{X}^*, t) = X_*^i$, with the reference configuration chosen to be the initial configuration. For the shifter tensor of the change of frame one gets

$$Q(\mathbf{x}^*, t) = \begin{bmatrix} 1 & \alpha t \\ 0 & 1 \end{bmatrix}; \quad Q^{-1}(\mathbf{x}, t) = \begin{bmatrix} 1 & -\alpha t \\ 0 & 1 \end{bmatrix}, \quad (1.91)$$

and the local curvilinear contravariant and covariant bases reads

$$\begin{aligned} \vec{g}_1 &= \vec{e}_1; & \vec{g}_2 &= \alpha t \vec{e}_1 + \vec{e}_2; & \underline{g}^1 &= \underline{e}^1 - \alpha t \underline{e}^2; & \underline{g}^2 &= \underline{e}^2 \\ \vec{e}_1 &= \vec{g}_1; & \vec{e}_2 &= \alpha t \vec{g}_1 - \vec{g}_2; & \underline{e}^1 &= \underline{g}^1 + \alpha t \underline{g}^2; & \underline{e}^2 &= \underline{g}^2 \end{aligned} \quad (1.92)$$

Inserting for $\vec{e}_i$ from eq. (1.92) into the solution (1.90) gives the components of the solution in the convected frame ϕ^* :

$$\vec{a}(\mathbf{x}^*=0, t) = a_0^{11} \vec{g}_1 \otimes \vec{g}_1 + a_0^{12} \vec{g}_1 \otimes \vec{g}_2 + a_0^{12} \vec{g}_2 \otimes \vec{g}_1 + a_0^{22} \vec{g}_2 \otimes \vec{g}_2. \quad (1.93)$$

We see that the components $a_{\phi^*}^{ij}$ of the solution are constant in time. Here we have shown it only at the origin $\mathbf{x}^*=0$, but it is valid for any particle of the body.

It is instructive to analyze the geometrical meaning of the fact that $a_{\phi^*}^{ij}$ are constant in time. Any contravariant tensor field $\vec{a}$ is a mapping from the dual space $\mathcal{L}(\mathbb{V})$ to $\mathbb{V}$: $\vec{a}(\underline{n}) = \vec{w}$; $\underline{n} \in \mathcal{L}(\mathbb{V})$, $\vec{w} \in \mathbb{V}$. Generally, it is a temporally and spatially dependent mapping. Let us consider a scalar field s and a set of parametrized curves with tangents given by a vector field $\vec{w}$, such that $\vec{a}(\nabla s) = \vec{w}$

holds at every point p at time $t=0$. Let us further assume that the scalar field s and the parametrized curves are frozen into the body, that is, $s(p, t)=s(p)$ and $w_{\phi^*}^i(p, t)=w_{\phi^*}^i(p)$. Then, the time-constancy of $a_{\phi^*}^{ij}$ implies that $\overleftrightarrow{a}(\nabla s) = \overleftrightarrow{w}$ holds at every point p and every time t throughout the motion of the body.

To inspect the meaning of the Jaumann rate, let us assume a different contravariant tensor $\overleftrightarrow{b}(\mathbf{x}, t) = b_{\phi}^{ij}(\mathbf{x}, t) \vec{e}_i \otimes \vec{e}_j$, whose temporal evolution is governed by

$$\frac{\mathcal{D}_{\text{Jau}} b_{\phi}^{ij}}{\mathcal{D}t} = \frac{\mathcal{D} b_{\phi}^{ij}}{\mathcal{D}t} - \mathbb{W}_{\phi}^{ik} b_{\phi}^{kj} + b_{\phi}^{ik} \mathbb{W}_{\phi}^{kj} = 0. \quad (1.94)$$

At the origin of the frame eq. (1.94) reduces to

$$\frac{\partial}{\partial t} \begin{bmatrix} b_{\phi}^{11} & b_{\phi}^{12} \\ b_{\phi}^{12} & b_{\phi}^{22} \end{bmatrix} = \begin{bmatrix} \alpha b_{\phi}^{12} & \frac{\alpha}{2}(b_{\phi}^{22} - b_{\phi}^{11}) \\ \frac{\alpha}{2}(b_{\phi}^{22} - b_{\phi}^{11}) & -\alpha b_{\phi}^{12} \end{bmatrix},$$

which has the following solution

$$\begin{aligned} \overleftrightarrow{b}(\mathbf{x}=0, t) &= [c_1 \sin(\alpha t) - c_2 \cos(\alpha t) + c_3] \vec{e}_1 \otimes \vec{e}_1 + [c_1 \cos(\alpha t) + c_2 \sin(\alpha t)] \vec{e}_1 \otimes \vec{e}_2 \\ &+ [c_1 \cos(\alpha t) + c_2 \sin(\alpha t)] \vec{e}_2 \otimes \vec{e}_1 - [c_1 \sin(\alpha t) - c_2 \cos(\alpha t) - c_3] \vec{e}_2 \otimes \vec{e}_2, \end{aligned} \quad (1.95)$$

where c_1, c_2, c_3 are constants that can be determined from the initial conditions $b_{\phi}^{ij}(\mathbf{x}=0, t=0) = b_0^{ij}$.

Again, there exists a frame ϕ^* in which the components of $\overleftrightarrow{b}$ are time constant. It is the corotated frame, defined as

$$\left. \begin{aligned} x^1(\mathbf{x}^*, t) &= \sqrt{(x_*^1)^2 + (x_*^2)^2} \cos\left(\frac{\alpha}{2}t\right) \\ x^2(\mathbf{x}^*, t) &= \sqrt{(x_*^1)^2 + (x_*^2)^2} \sin\left(\frac{\alpha}{2}t\right) \end{aligned} \right\} \quad \left. \begin{aligned} x_*^1(\mathbf{x}, t) &= \sqrt{(x^1)^2 + (x^2)^2} \cos\left(\frac{\alpha}{2}t\right) \\ x_*^2(\mathbf{x}, t) &= -\sqrt{(x^1)^2 + (x^2)^2} \sin\left(\frac{\alpha}{2}t\right) \end{aligned} \right\}.$$

Unlike in the case of the convected frame, the corotated frame is related to the Cartesian frame ϕ via Euclidean change of frame, but it is only because we are studying homogeneous simple shearing here. The relative motion of both frames is a rotation with the angular speed of $\alpha/2$ (compare with the spin of the studied motion). For the shifter tensor we get

$$Q(\mathbf{x}^*, t) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}t\right) & \sin\left(\frac{\alpha}{2}t\right) \\ -\sin\left(\frac{\alpha}{2}t\right) & \cos\left(\frac{\alpha}{2}t\right) \end{bmatrix}; \quad Q^{-1}(\mathbf{x}, t) = \begin{bmatrix} \cos\left(\frac{\alpha}{2}t\right) & -\sin\left(\frac{\alpha}{2}t\right) \\ \sin\left(\frac{\alpha}{2}t\right) & \cos\left(\frac{\alpha}{2}t\right) \end{bmatrix},$$

confirming that the change of frame is a time dependent rotation, because the inverse of the shifter tensor is its transpose: $Q^{-1}(\mathbf{x}, t) = Q^T(t)$. The local curvi-

linear contravariant and covariant bases are related via

$$\begin{aligned}\vec{g}_1 &= \cos\left(\frac{\alpha}{2}t\right)\vec{e}_1 - \sin\left(\frac{\alpha}{2}t\right)\vec{e}_2; & \vec{g}_2 &= \cos\left(\frac{\alpha}{2}t\right)\vec{e}_2 + \sin\left(\frac{\alpha}{2}t\right)\vec{e}_1 \\ \underline{g}^1 &= \cos\left(\frac{\alpha}{2}t\right)\underline{e}^1 - \sin\left(\frac{\alpha}{2}t\right)\underline{e}^2; & \underline{g}^2 &= \cos\left(\frac{\alpha}{2}t\right)\underline{e}^2 + \sin\left(\frac{\alpha}{2}t\right)\underline{e}^1.\end{aligned}\tag{1.96}$$

Since both frames are orthonormal, the transformation properties of one-forms and vectors are the same. With the help of eq. (1.96) the solution (1.95) can be, after some algebraic manipulations, written in terms of its components in the corotated frame ϕ^* :

$$\overrightarrow{b}(\mathbf{x}^*=0, t) = (c_3 - c_2)\vec{g}_1 \otimes \vec{g}_1 + c_1\vec{g}_1 \otimes \vec{g}_2 + c_1\vec{g}_2 \otimes \vec{g}_1 + (c_3 + c_2)\vec{g}_2 \otimes \vec{g}_2, \tag{1.97}$$

which are indeed constant in time.

We conclude the example with two remarks. First, the local covariant and contravariant bases are indistinguishable in the corotated frame, cf. eq. (1.96), and so the calculations with the contravariant tensor field $\overrightarrow{b}$ would be the same for second-order tensors of any kind, namely for any left covariant mixed tensor field, right covariant mixed tensor field, and covariant tensor field. All the components of tensor fields are constant in the corotated frame when their temporal evolution is governed by eq. (1.94). For the convected frame and the upper convected derivative this is different: only the contravariant components of tensors are constant in the convected frame when their temporal evolution is governed by eq. (1.89) (see also the discussion below eq. (1.46) in the main text).

Second, it is instructive to repeat the procedure with the convected frame for the case of simple stretching instead of simple shearing. Simple stretching can be defined as $\chi_{\kappa, \phi}^1(\mathbf{X}, t) = X_1 \exp(\alpha_1 t)$; $\chi_{\kappa, \phi}^2(\mathbf{X}, t) = X_2 \exp(\alpha_2 t)$, and it can be shown that while setting the Oldroyd derivative of a contravariant tensor to zero leads to constancy of its components in the convected frame (they are constant along particle trajectories), setting the Truesdell rate to zero results in constant components in the Cartesian frame (again, constant along particle trajectories). In other words, the first solution satisfies $a_{\phi^*}^{ij}(\mathbf{x}^*, t) = a_{\phi^*}^{ij}(\mathbf{x}^*)$, while the latter solution can be satisfies $\overrightarrow{a}(p, t) = \overrightarrow{a}(p)$.