

Chapter 7

Foundations of the Theory of the Earth's Gravity Field

7.1 Basic Notions

Consider a particle, its mass being denoted by m . According to Newton's law of universal gravitation, the intensity of the gravitational field due to this particle at a distance r is of magnitude

$$E = \frac{Gm}{r^2} . \quad (7.1)$$

Introducing the radius vector, $\mathbf{r}$, which originates at the particle, the vector of intensity may be expressed as

$$\mathbf{E}(\mathbf{r}) = -\frac{Gm}{r^3} \mathbf{r} . \quad (7.2)$$

Its radial component, being considered positive away from the particle,

$$E_r = -\frac{Gm}{r^2} . \quad (7.3)$$

Let us introduce the gravitational potential, W , so that

$$\mathbf{E} = -\text{grad } W . \quad (7.4)$$

The radial component of intensity is then

$$E_r = -\frac{\partial W}{\partial r} , \quad (7.5)$$

and for the gravitational potential of a particle we have

$$W = -\frac{Gm}{r} . \quad (7.6)$$

However, in mathematics, astronomy, geodesy and geophysics, we often give the potential the opposite sign, i.e. we change the signs in formulae (7.4) to (7.6). The new function, $U = -W$, is called the potential function, or simply the

potential. Consequently, the *gravitational potential of a particle*, according to the new sign convention, reads

$$U = +\frac{Gm}{r} , \quad (7.7)$$

and the corresponding *intensity of the gravitational field* is

$$\mathbf{E} = +\text{grad} U . \quad (7.8)$$

In this chapter we shall use this sign convention.

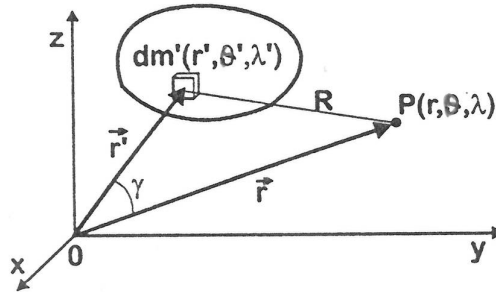


Figure 7.1: Notation describing the position of the observer, P , and of mass element dm .

Now, let us generalise these expressions for the gravitational field due to an arbitrary body (Fig. 7.1). Denote by P the position of the observer, and by dm the mass element of the body. Their radius-vectors will be denoted by $\mathbf{r}$ and $\mathbf{r}'$, respectively:

$$\mathbf{r} = (x, y, z) = (r \sin \theta \cos \lambda, r \sin \theta \sin \lambda, r \cos \theta) , \quad (7.9)$$

$$\mathbf{r}' = (x', y', z') = (r' \sin \theta' \cos \lambda', r' \sin \theta' \sin \lambda', r' \cos \theta') ,$$

where x, y, z and x', y', z' are the Cartesian coordinates, and $r, \theta, \lambda, r', \theta', \lambda'$ the spherical coordinates of these points. Using the *superposition principle*, the gravitational potential of the body as a whole may be expressed as

$$U(P) = G \iiint \frac{dm}{R} , \quad (7.10)$$

where the integration is carried out over all the mass elements of the body. This integral belongs to the category of Stieltjes integrals. Hence, the most general expression for the gravitational potential is given by Stieltjes integral (7.10).

However, we shall usually assume that the mass distribution within the body is sufficiently continuous to be able to introduce density, ρ . Then

$$dm = \rho dV, \quad (7.11)$$

where $\rho = \rho(x', y', z') = \rho(r', \theta', \lambda')$ and dV is the corresponding volume element. In this case, integral (7.10) simplifies to usual integral

$$U(P) = G \iiint_V \frac{\rho}{R} dV, \quad (7.12)$$

where the integration is carried out over the volume of the body, V , i.e. over primed coordinates x', y', z' , or r', θ', λ' . We shall usually express the gravitational potential in the general form (7.12). (Note that the density cannot be introduced at some discontinuities within the Earth, since the limits on the opposite sides are different.)

7.2 Gravity Field

Let us now describe the gravitational field of an arbitrary body in a rotation system.

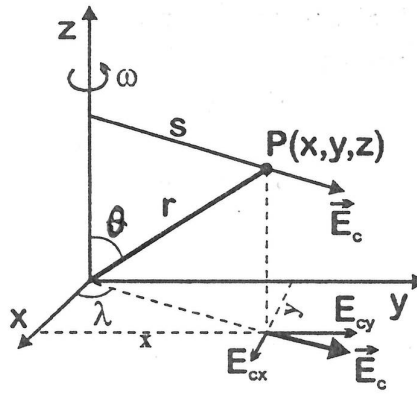


Figure 7.2: Decomposition of the centrifugal acceleration.

Assume that the body and the observer are fixed in a reference system which rotates about the z -axis at angular velocity ω (Fig. 7.2). The centrifugal acceleration at point P is $E_c = \omega^2 s$, where s is the distance from the rotation axis. From the similarity of the triangles in the xy -plane, it follows that

$$\frac{E_{cx}}{E_c} = \frac{x}{s}, \quad \frac{E_{cy}}{E_c} = \frac{y}{s},$$

where E_{cx} and E_{cy} are the components of E_c along axes x and y , respectively. By substituting $E_c = \omega^2 s$, we arrive at

$$\mathbf{E}_c = (\omega^2 x, \omega^2 y, 0) . \quad (7.13)$$

We are seeking the potential U_c of the centrifugal force, satisfying the relation

$$\mathbf{E}_c = \text{grad} U_c = \left(\frac{\partial U_c}{\partial x}, \frac{\partial U_c}{\partial y}, \frac{\partial U_c}{\partial z} \right) . \quad (7.14)$$

By comparing (7.13) and (7.14), we arrive at the potential of the centrifugal force,

$$U_c = \frac{1}{2} \omega^2 (x^2 + y^2) = \frac{1}{2} \omega^2 r^2 \sin^2 \theta . \quad (7.15)$$

The force acting at a fixed point P of a rotating system is the superposition of the gravitational and centrifugal forces. We have thus arrived at the concept of the *gravity field* as the superposition of the gravitational field and the field of the centrifugal force. Hence, we shall refer to the vector sum of the gravitational and centrifugal accelerations as the *gravity acceleration*. Analogously, the *gravity potential* is the sum of the gravitational and centrifugal potentials,

$$U(P) = G \iiint_V \frac{\rho}{R} dV + \frac{1}{2} \omega^2 (x^2 + y^2) . \quad (7.16)$$

This is the basic, general formula for the gravity potential.

7.3 Expansion of the External Gravity Potential into a Series of Spherical Harmonics

By substituting the expansion (6.24a) for $1/R$ into (7.16) and using spherical coordinates, we get

$$U(r, \theta, \lambda) = \frac{G}{r} \iiint_V \rho \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos \gamma) dV + \frac{1}{2} \omega^2 r^2 \sin^2 \theta , \quad (7.17)$$

where we integrate over the points of volume V , i.e. over coordinates r', θ' and λ' .

Assume that the distance r of the observer from origin O is so large that $r > r'$ for all points of the body (the observer is outside the sphere which has its centre at origin O and contains the whole body in its interior). The series in

(7.17) is then convergent independently of coordinates r', θ', λ' (uniform convergence) and the integration and summation may be interchanged. Then

$$U(r, \theta, \lambda) = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \lambda)}{r^{n+1}} + \frac{1}{2} \omega^2 r^2 \sin^2 \theta, \quad (7.18)$$

where

$$Y_n(\theta, \lambda) = G \iiint (r')^n P_n(\cos \gamma) dm, \quad (7.19)$$

and dm being used instead of ρdV .

Let us investigate a few low terms of the series. The first spherical function Y_0 has a very simple physical meaning:

$$Y_0 = G \iiint dm = GM, \quad (7.20)$$

where M is the total mass of the body.

To simplify the following terms, let us choose the coordinate system as follows:

1. The origin of the coordinate system, O , is at the centre of mass of the body, T . Consequently, the radius-vector of this centre is zero, i.e.

$$\mathbf{r}_T = \frac{1}{M} \iiint \mathbf{r}' dm = 0. \quad (7.21)$$

2. The axes of the coordinate system coincide with the principal axes of inertia. In this case, the deviatoric moments vanish.

Under these assumptions we shall calculate the other terms in series (7.18). For $n = 1$,

$$Y_1 = G \iiint r' \cos \gamma dm = G \iiint r' [\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\lambda - \lambda')] dm,$$

where we have substituted (5.37) for $\cos \gamma$. Thus

$$Y_1 = G \left\{ \cos \theta \iiint r' \cos \theta' dm + \sin \theta \cos \lambda \iiint r' \sin \theta' \cos \lambda' dm + \right. \\ \left. + \sin \theta \sin \lambda \iiint r' \sin \theta' \sin \lambda' dm \right\}.$$

The first integral in this formula, after dividing by M , represents the z -coordinate of the centre of mass. However, according to Assumption 1, this coordinate is equal to zero. Similarly, the remaining two integrals correspond to the remaining

two coordinates of the centre of mass, and are also equal to zero. Consequently, under Assumption 1,

$$Y_1(\theta, \lambda) = 0. \quad (7.22)$$

Since Y_1/r^2 represents the potential of a dipole, this means that the dipole component of the gravitational potential vanishes if the origin of the coordinate system is put at the centre of mass.

We shall see that the coefficients in function Y_2 can be expressed in terms of the moments of inertia of the body. We have had (see Section 6.9)

$$P_2(\cos \gamma) = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) + \quad (a)$$

$$+ 3 \sin \theta \cos \theta \sin \theta' \cos \theta' \cos \lambda \cos \lambda' + \quad (b)$$

$$+ 3 \sin \theta \cos \theta \sin \theta' \cos \theta' \sin \lambda \sin \lambda' + \quad (c)$$

$$+ \frac{3}{4} \sin^2 \theta \sin^2 \theta' \cos 2\lambda \cos 2\lambda' + \quad (d)$$

$$+ \frac{3}{4} \sin^2 \theta \sin^2 \theta' \sin 2\lambda \sin 2\lambda', \quad (e)$$

where the individual parts of this formula have been denoted by the letters a to e .

It follows from Assumption 2 that the deviatoric moments vanish:

$$0 = J_{xy} = \iiint x'y' \, dm = \iiint r'^2 \sin^2 \theta' \cos \lambda' \sin \lambda' \, dm,$$

$$0 = J_{xz} = \iiint x'z' \, dm = \iiint r'^2 \sin \theta' \cos \theta' \cos \lambda' \, dm,$$

$$0 = J_{yz} = \iiint y'z' \, dm = \iiint r'^2 \sin \theta' \cos \theta' \sin \lambda' \, dm.$$

By comparing these integrands with the expressions in e , b , c , it can be seen that the contributions of these parts to Y_2 will vanish.

It remains to express the contributions to Y_2 coming from parts a and d . Denote by A , B , C the moments of inertia about axes x , y , z , respectively. Then

$$A = \iiint (y'^2 + z'^2) \, dm = \iiint r'^2 (\sin^2 \theta' \sin^2 \lambda' + \cos^2 \theta') \, dm,$$

$$B = \iiint (x'^2 + z'^2) dm = \iiint r'^2 (\sin^2 \theta' \cos^2 \lambda' + \cos^2 \theta') dm ,$$

$$C = \iiint (x'^2 + y'^2) dm = \iiint r'^2 \sin^2 \theta' dm = \iiint r'^2 (1 - \cos^2 \theta') dm .$$

Consequently,

$$A + B = \iiint r'^2 (\sin^2 \theta' + 2 \cos^2 \theta') dm = \iiint r'^2 (1 + \cos^2 \theta') dm ,$$

$$\frac{A+B}{2} - C = \iiint r'^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) dm ;$$

see the expression in part *a*. The difference

$$B - A = \iiint r'^2 \sin^2 \theta' (\cos^2 \lambda' - \sin^2 \lambda') dm$$

yields the contribution coming from part *d*. Hence,

$$Y_2 = G \left(\frac{A+B}{2} - C \right) \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + G \frac{3}{4} (B - A) \sin^2 \theta \cos 2\lambda . \quad (7.23)$$

The gravity potential (7.18) may now be expressed as

$$\begin{aligned} U(r, \theta, \lambda) = & \frac{GM}{r} + \frac{G}{2r^3} \left(C - \frac{A+B}{2} \right) (1 - 3 \cos^2 \theta) + \\ & + \frac{3}{4} \frac{G}{r^3} (B - A) \sin^2 \theta \cos 2\lambda + \frac{1}{2} \omega^2 r^2 \sin^2 \theta + T(r, \theta, \lambda) , \end{aligned} \quad (7.24)$$

where

$$T(r, \theta, \lambda) = \sum_{n=3}^{\infty} \frac{Y_n(\theta, \lambda)}{r^{n+1}} . \quad (7.25)$$

Term *T* will be referred to as the *disturbing potential*.

Let us introduce geocentric latitude $\phi = 90^\circ - \theta$, and measure angle λ from the Greenwich meridian, not from the principal axis. Denoting by λ_0 the longitude of the principal axis (i.e. $\lambda_0 = 14.9^\circ \text{ W}$), we must replace λ in (7.24) by $(\lambda - \lambda_0)$:

$$\begin{aligned}
U(r, \phi, \lambda) = & \frac{GM}{r} + \frac{G}{2r^3} \left(C - \frac{A+B}{2} \right) (1 - 3 \sin^2 \phi) + \\
& + \frac{3}{4} \frac{G}{r^3} (B - A) \cos^2 \phi \cos 2(\lambda - \lambda_0) + \frac{1}{2} \omega^2 r^2 \cos^2 \phi + T(r, \phi, \lambda) .
\end{aligned} \tag{7.26}$$

The first term on the right-hand side of the latter formula, i.e. GM/r , yields the potential of a spherically symmetric body. This is the main term in the expansion of the Earth's gravity field. We shall see that the remaining terms are three or more orders smaller. The second term contains the difference between the polar and equatorial moments of inertia, so that this term may be expected to be closely related to the flattening of the Earth; we shall prove this in Section 7.4. The next term, containing the difference $B - A$, describes the influence of the equatorial flattening. However, in the case of the Earth, this term is much smaller than the preceding one and, therefore, this term is very often neglected. The fourth term describes the centrifugal potential, and term T describes the influence of other irregularities in the density distribution of the body.

To simplify formula (7.26), let us assume that $B = A$ (actually, we may consider different values of A and B , but we transfer the corresponding small term to disturbing potential T). Then

$$\boxed{U(r, \phi, \lambda) = \frac{GM}{r} + \frac{G}{2r^3} (C - A) (1 - 3 \sin^2 \phi) + \frac{1}{2} \omega^2 r^2 \cos^2 \phi + T(r, \phi, \lambda)} . \tag{7.27}$$

This form of the gravity potential is used in many applications. In addition to the leading term, GM/r , it explicitly contains the contributions due to the polar flattening and the non-inertiality of the reference system, whereas the other irregularities are formally hidden in disturbing term T . Formally, we shall also express formula (7.27) as

$$U(r, \phi, \lambda) = V(r, \phi) + T(r, \phi, \lambda) , \tag{7.28}$$

where term V represents the first three terms on the right-hand side of (7.27). We shall refer to term V as the *normal potential* or regular potential.

In some applications, e.g., in satellite geodesy, we need explicit expressions also for disturbing potential T . This is feasible if the addition theorem for Legendre polynomials is used in all terms Y_n , see (7.19) and (6.30):

$$Y_n(\theta, \lambda) = \sum_{m=0}^n [A_n^m \cos m\lambda + B_n^m \sin m\lambda] P_n^m(\cos \theta) , \tag{7.29}$$

where

$$A_n^m = G \epsilon_m \frac{(n-m)!}{(n+m)!} \iiint (r')^n P_n^m(\cos \theta') \cos m\lambda' d m , \quad (7.30)$$

$$B_n^m = G \epsilon_m \frac{(n-m)!}{(n+m)!} \iiint (r')^n P_n^m(\cos \theta') \sin m\lambda' d m ,$$

and $\epsilon_0 = 1, \epsilon_1 = \epsilon_2 = \dots = 2$. If the density distribution within the body were known, we could determine coefficients A_n^m and B_n^m by calculating integrals (7.30). This is not the case of the real Earth. However, these coefficients can be determined directly from perturbations of satellite orbits and, partly, from gravity measurements on the surface of the Earth.

Considering the leading role of the first term, $A_0^0 = GM$, and the choice of the origin of the coordinate system at the mass centre, gravity potential (7.18) may also be expressed as

$$U(r, \theta, \lambda) = \frac{GM}{r} \left\{ 1 + \sum_{n=2}^{\infty} \left(\frac{a}{r} \right)^n \sum_{m=0}^n [J_n^m \cos m\lambda + S_n^m \sin m\lambda] P_n^m(\cos \theta) \right\} + \frac{1}{2} \omega^2 r^2 \sin^2 \theta , \quad (7.31)$$

where a is a length factor, usually the equatorial radius of the Earth. In this case, coefficients J_n^m and S_n^m are dimensionless,

$$J_n^m = \frac{A_n^m}{GMa^n} , \quad S_n^m = \frac{B_n^m}{GMa^n} .$$

Note that several other variants of (7.31) are also used in the geophysical literature.

7.4 Equipotential Surfaces. The Geoid and Spheroid

7.4.1 Equation of the geoid

The expressions in the previous section make it possible to define the figure of the Earth physically as a figure of a selected equipotential surface of the gravity field. Note that, in this situation, Gauss and Bessel spoke of the mathematical figure of the Earth; see Chapter 1.

The equation

$$U(r, \phi, \lambda) = c_0$$

determines a system of equipotential surfaces of the gravity field if constant c_0 takes different values. The most important among these surfaces is the equipotential surface which is closest to the mean level of the seas. This equipotential surface is called the *geoid*. By mean sea level we understand the level from which all fluctuations have been removed, i.e. the fluctuations due to waves, tides and some other sources. We shall denote the corresponding constant for the geoid by U_0 . Thus, the equation

$$\boxed{U(r, \phi, \lambda) = U_0} \quad (7.32)$$

represents the *equation of the geoid*. A recent value of the geoidal potential is (Burša et al., 1995)

$$U_0 = (62\,636\,857.0 \pm 1.0) \text{ m}^2 \text{ s}^{-2} . \quad (7.33)$$

From the physical point of view, we shall take the figure of the geoid to be the figure of the Earth. In other words, by the figure of the Earth we shall understand the figure of the geoid.

Note that the gradient of the potential yields the gravity acceleration, $\mathbf{g}$:

$$\mathbf{g} = \text{grad } U . \quad (7.34)$$

7.4.2 Equation of the spheroid

The figure of the geoid is very complicated and, therefore, cannot be simply expressed mathematically. Hence, we shall approximate the geoid by the *spheroid*, by neglecting the disturbing term. Thus, using the notation in (7.28), the *equation of the spheroid* can formally be expressed as

$$\boxed{V(r, \phi, \lambda) = U_0} , \quad (7.35)$$

where U_0 is the same constant as in (7.32). The corresponding acceleration

$$\boldsymbol{\gamma} = \text{grad } V \quad (7.36)$$

is called the *normal gravity acceleration* or *normal gravity*. Note that the spheroid is not an equipotential surface of the gravity field, but only of its normal (regular) part.

Other approximations of the geoid are also used, such as the tri-axial ellipsoid, ellipsoid of revolution, or a sphere as the simplest approximation.

We shall now derive an explicit form of the equation of the spheroid (7.35). Using (7.27) the equation of the spheroid may be expressed as