

3.2 Equation of Motion of a Particle in a Non-Inertial System

Again consider two coordinate system and assume that:

1. The fixed system is inertial (for Newton's second law to hold true);
2. The origins of the systems are generally different.

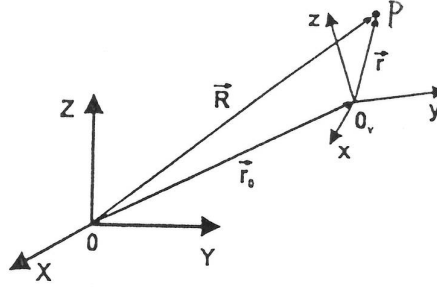


Figure 3.2: Position of point P in the two systems: $\mathbf{R}$ and $\mathbf{r}$ are the radius-vectors in the inertial and non-inertial systems, respectively.

Denote the Cartesian axes in the fixed system X, Y, Z , and in the moving system x, y, z (Figure 3.2). Denote $\mathbf{r}_0$ the radius-vector of the origin of the moving system with respect to the fixed system. The position of an arbitrary point P can be described by two radius-vectors, by $\mathbf{R}$ in the fixed system, and by $\mathbf{r}$ in the moving system, where

$$\mathbf{R} = \mathbf{r}_0 + \mathbf{r} . \quad (3.9)$$

We shall need the second derivative of this vector. In the first derivative,

$$\dot{\mathbf{R}} = \dot{\mathbf{r}}_0 + \dot{\mathbf{r}} , \quad (3.10)$$

the terms on the right-hand side could be expressed by means of Resal's theorem (3.7) as follows:

$$\dot{\mathbf{r}}_0 = \frac{d\mathbf{r}_0}{dt} + \boldsymbol{\omega} \times \mathbf{r}_0 = \mathbf{v}_0 , \quad (3.11)$$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} . \quad (3.12)$$

However, we shall not usually apply Resal's theorem to $\dot{\mathbf{r}}_0$ (we shall not describe the time variation of $\mathbf{r}_0$ from the moving system, as this would usually

be impractical). We shall simply write $\dot{\mathbf{r}}_0 = \mathbf{v}_0$, where $\mathbf{v}_0$ is the absolute velocity of the origin of the non-inertial system. Consequently,

$$\boxed{\dot{\mathbf{R}} = \mathbf{v}_0 + \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}} \quad (3.13)$$

In calculating the second derivative of $\mathbf{R}$ we shall apply Resal's theorem to all terms of (3.13):

$$\begin{aligned} \ddot{\mathbf{R}} &= \frac{d}{dt} \left[\mathbf{v}_0 + \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \right] + \boldsymbol{\omega} \times \left[\mathbf{v}_0 + \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \right] = \\ &= \frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{v}_0}{dt} + \boldsymbol{\omega} \times \mathbf{v}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) . \end{aligned}$$

Newton's second law in the inertial system,

$$m\ddot{\mathbf{R}} = \mathbf{F} , \quad (3.14)$$

then yields the equation of motion in the non-inertial system in the following form:

$$\boxed{m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - 2m\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} - m \left[\frac{d\mathbf{v}_0}{dt} + \boldsymbol{\omega} \times \mathbf{v}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \right]} \quad (3.15)$$

The latter equation resembles the equation of motion in an inertial system (3.14), but in addition to the true force $\mathbf{F}$, there are two other terms, the so-called fictitious forces, on the right-hand side of the equation of motion:

$$\boxed{m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + \mathbf{F}_c + \mathbf{F}_d} \quad (3.16)$$

The force

$$\mathbf{F}_c = -2m\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = 2m\mathbf{v}_r \times \boldsymbol{\omega}$$

is called the Coriolis force, $\mathbf{v}_r = \frac{d\mathbf{r}}{dt}$ being the velocity relative to the non-inertial system. Force $\mathbf{F}_d$, representing the last term in equation (3.15), is called the drifting force.

Note that the Coriolis force disappears if:

$$1) \frac{d\mathbf{r}}{dt} = 0 \quad (\mathbf{v}_r = 0), \text{ or}$$

2) $\frac{d\mathbf{r}}{dt}$ is parallel to ω ($\mathbf{v}_r$ is parallel to ω).

Consider a very important special case when both the origins of the coordinate systems coincide, $O = O_v$. Then $\mathbf{v}_0 = 0$ and the equation of motion (3.15) simplifies to

$$\boxed{m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F} - 2m\omega \times \frac{d\mathbf{r}}{dt} - m\dot{\omega} \times \mathbf{r} - m\omega \times (\omega \times \mathbf{r})} \quad (3.17)$$

The last term but one in the latter equation is called the Euler force,

$$\mathbf{F}_e = -m\dot{\omega} \times \mathbf{r} ,$$

but this term disappears if the angular velocity of rotation ω is a constant vector. In considering mechanical phenomena on the rotating Earth, this term is usually neglected, as it is smaller than the other terms. Nevertheless, some changes of the angular velocity of the Earth's rotation will be discussed in the next chapter.

The last term in equation (3.17) represents the centrifugal force, which can easily be verified as follows. From the vector relation

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

it follows that

$$\omega \times (\omega \times \mathbf{r}) = \omega(\omega \cdot \mathbf{r}) - \mathbf{r}(\omega \cdot \omega) = \omega^2 \left[\frac{\omega}{\omega} \left(\frac{\omega}{\omega} \cdot \mathbf{r} \right) - \mathbf{r} \right] = -\rho \omega^2 .$$

Since $\frac{\omega}{\omega}$ is the unit vector at the rotation axis, term $\frac{\omega}{\omega} \left(\frac{\omega}{\omega} \cdot \mathbf{r} \right)$ is the projection of $\mathbf{r}$ onto this axis (Fig. 3.3). Consequently, ρ is the oriented distance from the rotation axis, and $-\omega \times (\omega \times \mathbf{r}) = +\omega^2 \rho$ is the centrifugal acceleration.

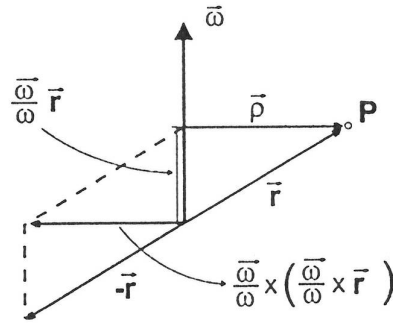


Figure 3.3: Explanation of the formula for the centrifugal acceleration.

1.8.2. Acceleration in co-rotation basis

- position vector $\mathbf{r}$
- velocity in IF: $\mathbf{v}|_{\text{IF}} := \frac{d\mathbf{r}}{dt}|_{\text{IF}}$
- velocity in coF: $\mathbf{v}|_{\text{coF}} := \frac{dr'_i}{dt}\mathbf{e}'_i := \frac{d\mathbf{r}}{dt}|_{\text{coF}}$
- relationship between the velocity in IF and coF

$$\mathbf{v}|_{\text{IF}} = \frac{d\mathbf{r}}{dt}|_{\text{coF}} + \boldsymbol{\omega} \times \mathbf{r} \quad (1.9)$$

- relationship between the acceleration in IF and coF

$$\begin{aligned} \mathbf{a}_{\text{IF}} &:= \frac{d\mathbf{v}_{\text{IF}}}{dt}|_{\text{IF}} = \frac{d\mathbf{v}_{\text{IF}}}{dt}|_{\text{coF}} + \boldsymbol{\omega} \times \mathbf{v}_{\text{IF}} = \\ &= \frac{d\mathbf{v}_{\text{coF}}}{dt}|_{\text{coF}} + \frac{d(\boldsymbol{\omega} \times \mathbf{r})}{dt}|_{\text{coF}} + \boldsymbol{\omega} \times (\mathbf{v}|_{\text{coF}} + \boldsymbol{\omega} \times \mathbf{r}) = \\ &= \frac{d\mathbf{v}_{\text{coF}}}{dt}|_{\text{coF}} + \underbrace{\dot{\boldsymbol{\omega}} \times \mathbf{r}}_{\text{Euler acc.}} + \underbrace{2\boldsymbol{\omega} \times \mathbf{v}_{\text{coF}}}_{\text{Coriolis acc.}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifug. acc.}} \end{aligned} \quad (1.10)$$

1.8.3. Euler's kinematical equations

- needed in order to describe rotation variation in space, relating rotation vector and time-derivative of Euler angles
- rotation matrices $\mathbf{D}$, $\mathbf{C}$ and $\mathbf{B}$ characterizing rotation (see definitions of the Euler angles), using right-hand side convention (cf., Van Hoolst, 2007)

- **precession** ψ , transformation matrix $\mathbf{D}$: $\mathbf{x}' = \mathbf{D} \cdot \mathbf{x}$, $\mathbf{D} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\omega}_D = (0, 0, \dot{\psi})$,

- **obliquity** ϵ , transformation matrix $\mathbf{C}$: $\mathbf{x}'' = \mathbf{C} \cdot \mathbf{x}' = \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{x}$, $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \epsilon & \sin \epsilon \\ 0 & -\sin \epsilon & \cos \epsilon \end{pmatrix}$, $\boldsymbol{\omega}_C = (\dot{\epsilon}, 0, 0)$

- **rotation** ϕ , transformation matrix $\mathbf{B}$: $\mathbf{x}''' = \mathbf{B} \cdot \mathbf{x}'' = \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{x}$, $\mathbf{B} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\omega}_B = (0, 0, \dot{\phi})$

- The above helps us to express the relation between the components (in the body frame) of the rotation vector and the time derivatives (in inertial space) of the Euler angles (Euler's kinematical equations), see, e.g. Van Hoolst (2007)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_B + \mathbf{B} \cdot \boldsymbol{\omega}_C + \mathbf{B} \cdot \mathbf{C} \cdot \boldsymbol{\omega}_D \quad (1.11)$$

in components

$$\omega_1 = \dot{\epsilon} \cos \phi + \dot{\psi} \sin \epsilon \sin \phi \quad (1.12)$$

$$\omega_2 = -\dot{\epsilon} \sin \phi + \dot{\psi} \sin \epsilon \cos \phi \quad (1.13)$$

$$\omega_3 = \dot{\phi} + \dot{\psi} \cos \epsilon. \quad (1.14)$$

1.8.4. Dynamics of a rigid body

- angular momentum

$$\mathbf{L} = \int_{\Omega} \underbrace{\mathbf{r} \times \mathbf{v}}_{\text{inertial frame}} dm = \left| \mathbf{v} \right|_{\text{IF}} = \underbrace{\mathbf{v}}_{=0 \text{ (rigid body)} \mid_{\text{coF}}} + \boldsymbol{\omega} \times \mathbf{r} = \int_{\Omega} \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm \quad (1.15)$$

- projection into general direction ζ

$$\mathbf{I}(\zeta, \boldsymbol{\omega}) := \mathbf{L} \cdot \zeta = \int_{\Omega} (\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})) \cdot \zeta dm = \int_{\Omega} (\zeta \times \boldsymbol{\omega}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) dm \quad (1.16)$$

- special case $\zeta = \boldsymbol{\omega}$

$$\mathbf{I}(\boldsymbol{\omega}, \boldsymbol{\omega}) = \int_{\Omega} \mathbf{v} \cdot \mathbf{v} dm = 2T, \quad (1.17)$$

where T is the kinetic energy.

- After additional manipulation, we can define the moment of inertia tensor $\mathbf{I}$

$$\mathbf{I}(\zeta, \boldsymbol{\omega}) = \mathbf{I}(\zeta_i \mathbf{e}_i, \omega_j \mathbf{e}_j) = \zeta_i \omega_j \mathbf{I}(\mathbf{e}_i, \mathbf{e}_j) = I_{ij} \zeta_i \omega_j. \quad (1.18)$$

- Using yet another modifications, we can write

$$I_{ij} = \int_{\Omega} (\mathbf{e}_i \times \mathbf{r}) \cdot (\mathbf{e}_j \times \mathbf{r}) dm = \quad (1.19)$$

$$= \int_{\Omega} (\mathbf{e}_i \times \mathbf{e}_k) \cdot (\mathbf{e}_j \times \mathbf{e}_l) x_k x_l dm = \quad (1.20)$$

$$= \int_{\Omega} \varepsilon_{rik} \varepsilon_{rjl} x_k x_l dm = \quad (1.21)$$

$$= \int_{\Omega} (\delta_{ij} x_k x_k - x_i x_j) dm \quad (1.22)$$

or using the invariant form

$$\boxed{\mathbf{I} = \int_{\Omega} (\mathbf{r} \cdot \mathbf{r} \mathbf{I} - \mathbf{r} \mathbf{r}) dm} \quad (1.23)$$

1.8.5. Euler's rotation equations

- Derivation from Conservation of angular momentum $\frac{d\mathbf{L}}{dt} = \mathbf{M}$, where $\mathbf{M}$ is the external moment (torque) and time derivative in the inertial frame.
- The above frame and derivative in body-frame lead to Euler's rotation equations:

$$\boxed{\left. \frac{d\mathbf{L}}{dt} \right|_{\text{IF}} = \left. \frac{d\mathbf{L}}{dt} \right|_{\text{coF}} + \boldsymbol{\omega} \times \mathbf{L}} \quad (1.24)$$