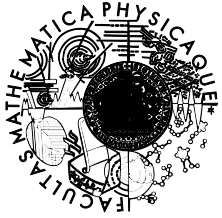


# Role of viscoelasticity in mantle convection models



Vojtěch Patočka<sup>1,2</sup>, Ondřej Čadek<sup>1</sup>, Paul Tackley<sup>2</sup>  
(patocka.vojtech@gmail.com)

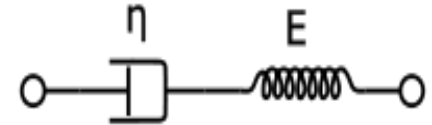
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Derivation of the model  
(objectivity vs. 3D analog)

Implementation  
(time step decoupling?)



$$\boldsymbol{\sigma} + t_M \overset{\nabla}{\boldsymbol{\sigma}} = 2\eta \mathbb{D}$$

$\boldsymbol{\sigma}$  ... Stress,  $t_M$  ... Maxwell relaxation time,  $\mathbb{D}$  ... Strain rate

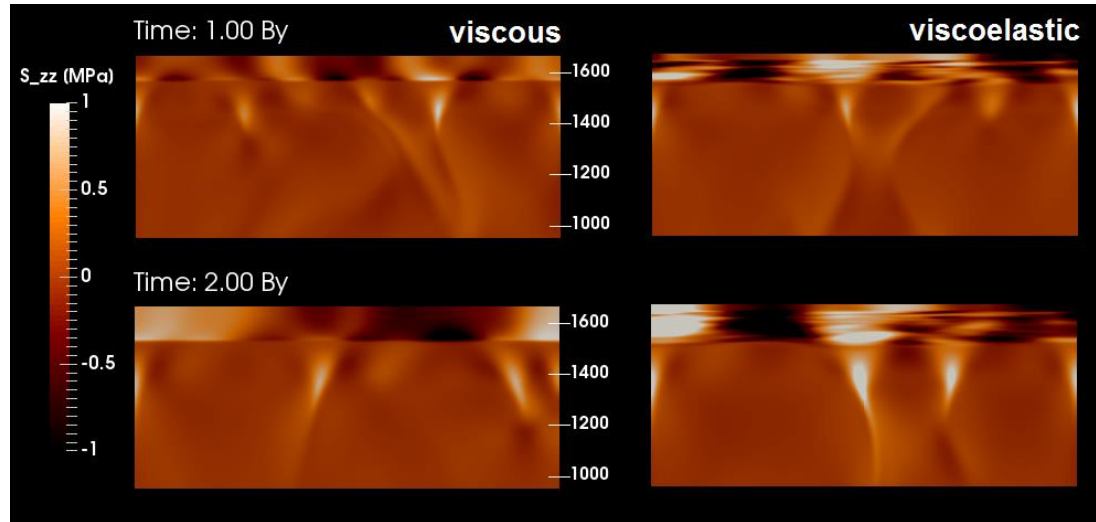
Viscoelastic material approximates the lithosphere better than a viscous fluid. We implement Maxwellian rheology into a mantle convection code StagYY, and focus on the new features in the resulting lithospheric stress distribution and topography.

Results:

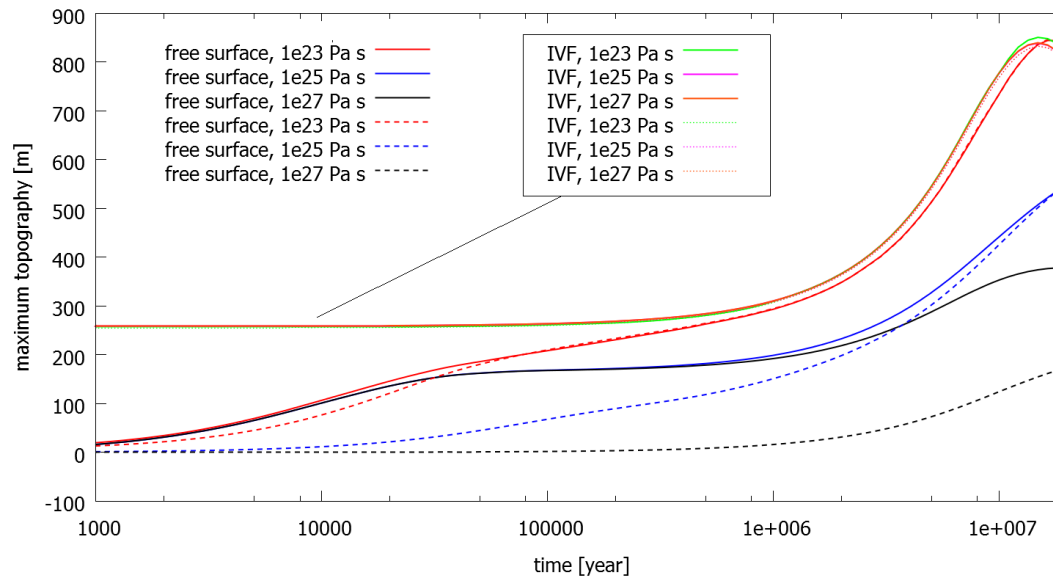
Viscoelastic  
Stress Memory

Topography  
(above a rising cylinder)

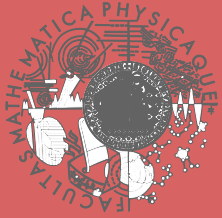
# Stress memory affects formation of the lithosphere



## Dynamical topography: high viscosity lids



# Role of viscoelasticity in mantle convection models



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$$\boldsymbol{\sigma} + \mathfrak{D} \overset{\nabla}{\boldsymbol{\sigma}} = 2\eta \mathbb{D} \quad \mathfrak{D}(p, T) := \tau_{\text{rel}} \frac{\kappa}{h^2} = \frac{\eta \kappa}{G h^2}$$

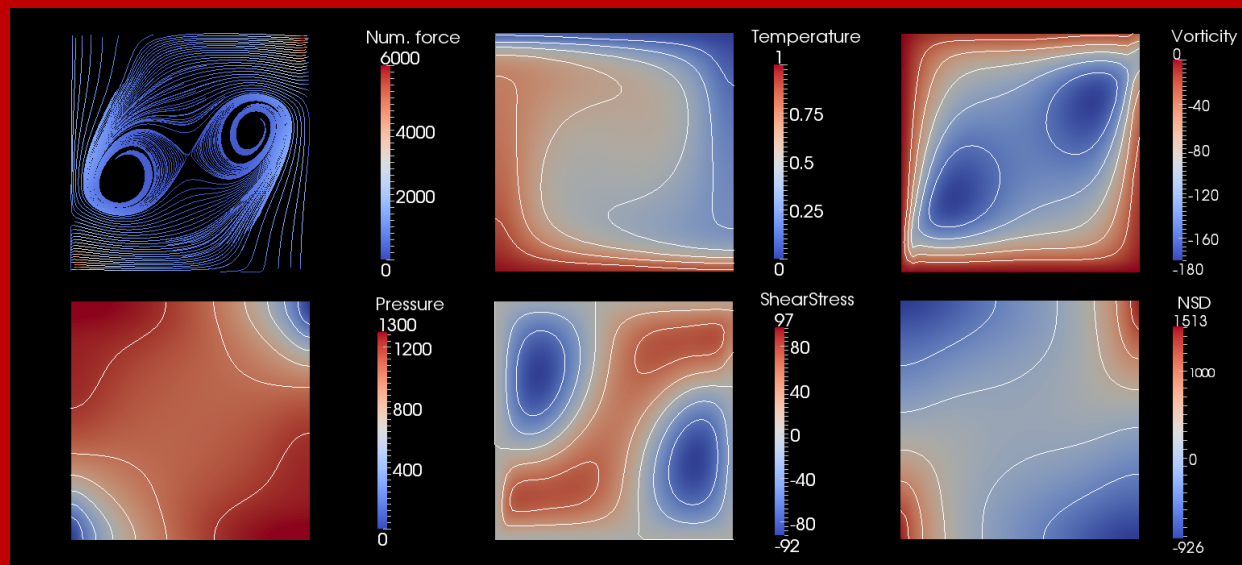
$$\overset{\nabla}{\boldsymbol{\sigma}} := \frac{\partial \boldsymbol{\sigma}}{\partial t} + \vec{v} \cdot \nabla \boldsymbol{\sigma} + (\boldsymbol{\sigma} \mathbb{W} - \mathbb{W} \boldsymbol{\sigma}) - \alpha (\mathbb{D} \boldsymbol{\sigma} + \boldsymbol{\sigma} \mathbb{D})$$

advection    co-rotation    co-deformation

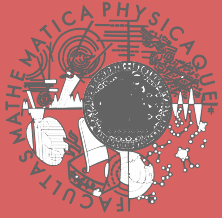
## Maxwellian viscoelasticity

Click on the equations to see how they can be discretized in order to be easily implemented into a viscous flow solver. Pictures below show stationary solutions of Boussinesq model.

Stationary flow with Jaumann rate (click on the picture to see the viscous case)



# Role of viscoelasticity in mantle convection models



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$$\boldsymbol{\sigma} + \mathfrak{D} \overset{\nabla}{\boldsymbol{\sigma}} = 2\eta \mathbb{D} \quad \mathfrak{D}(p, T) := \tau_{\text{rel}} \frac{\kappa}{h^2} = \frac{\eta \kappa}{G h^2}$$

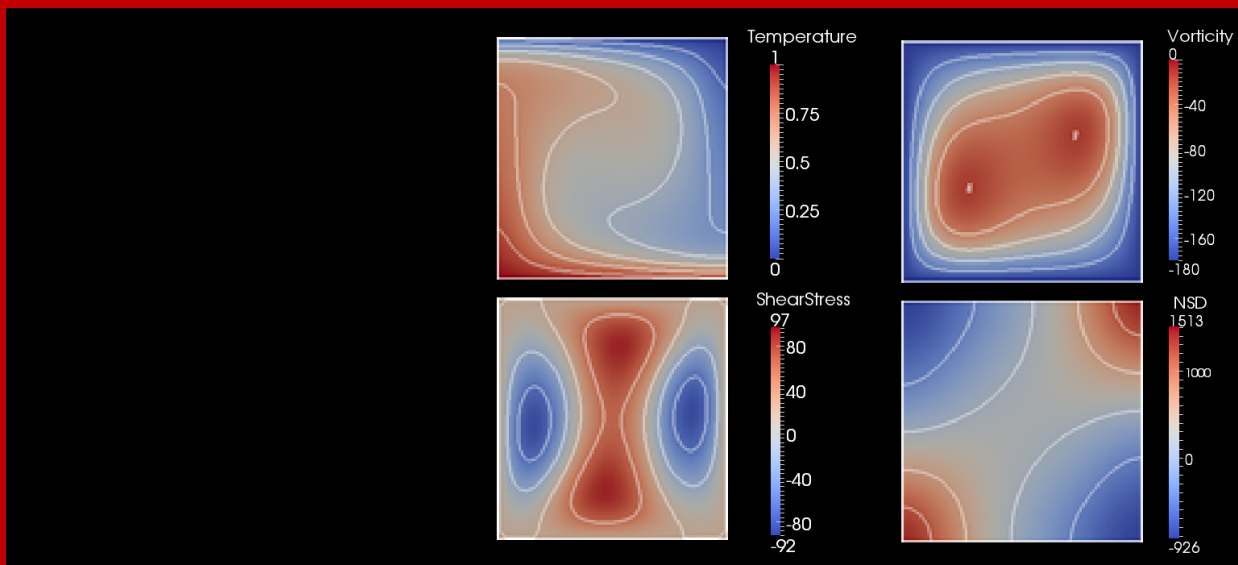
$$\overset{\nabla}{\boldsymbol{\sigma}} := \frac{\partial \boldsymbol{\sigma}}{\partial t} + \vec{v} \cdot \nabla \boldsymbol{\sigma} + (\boldsymbol{\sigma} \mathbb{W} - \mathbb{W} \boldsymbol{\sigma}) - \mathfrak{a}(\mathbb{D} \boldsymbol{\sigma} + \boldsymbol{\sigma} \mathbb{D})$$

advection    co-rotation    co-deformation

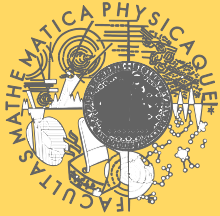
## Maxwellian viscoelasticity

Click on the equations to see how they can be discretized in order to be easily implemented into a viscous flow solver. Pictures below show stationary solutions of Boussinesq model.

Stationary flow with viscous rheology for comparison (click on figure to see Jaumann back)



# Role of viscoelasticity in mantle convection models



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Consider a loaded viscoelastic body, which has relaxation time huge compared to the observational period. Such body undergoes almost no observable deformation (strain rate close to zero), but it contains stored stresses.

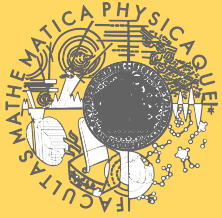
Now let us consider two observers of the body: One is attached to the body and the other one is moving around. If you require your constitutive equation to have the same form in all frames (i.e. the equation must look the same for both observers – like the continuity equation), then you are restricted in your choice of tensor fields' rates appearing in your equation.

If you choose only partial time derivative of stress in the rheological law, then you clearly get different stress field in the body when computed by the moving observer and different when computed by the fixed observer. This is unphysical (wrong) as it would result in a situation where the deformation of the body depends on which observer is looking at it. When the advective and corotational terms are included, both observers get the same stress field.

**Objectivity – Fundamental constraint**

| fixed                                                                                                                 | moving                                                                                                                                                                                                               |
|-----------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\nabla \cdot \boldsymbol{\tau} + \vec{f} = \rho \vec{a} \rightarrow 0$                                               | $\nabla \cdot \boldsymbol{\tau} + \vec{f} = \rho \vec{a} + \rho \vec{\omega} \times \vec{\omega} \times \vec{r} + 2\rho \vec{\omega} \times \vec{v} + \rho \frac{d\vec{\omega}}{dt} \times \vec{r} + \rho \vec{a}_0$ |
| $\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$                                                                    | $\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$                                                                                                                                                                   |
| $f(\boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t}, \mathbb{D}, \mathbb{W}, \vec{v}) = 0$        | $f(\boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t}, \mathbb{D}, \mathbb{W}, \vec{v}) = 0$                                                                                                       |
| e.g. $\boldsymbol{\sigma} + \tau_{\text{rel}} \overset{\nabla}{\boldsymbol{\sigma}} = 2\eta \mathbb{D} \rightarrow 0$ | $\boldsymbol{\sigma} + \tau_{\text{rel}} \overset{\nabla}{\boldsymbol{\sigma}} = 2\eta \mathbb{D} \rightarrow 0$                                                                                                     |

# Role of viscoelasticity in mantle convection models



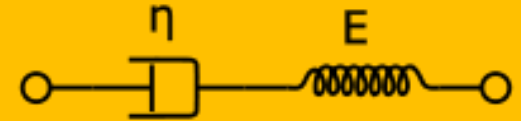
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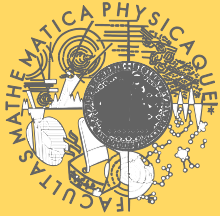
## 1D Analog – Linear viscoelasticity



- Maxwell element - spring and dashpot in series
- Forces equal:  $F = F_S = F_D$ , Deformation adds up:  $\Delta = \Delta_S + \Delta_D$
- Constitutive relation:  $\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = 2\mathbb{D}$ ,  $\mathbb{D}$ ... Strain rate
- Objectivity forms constraints on which tensor rates can be used in physically consistent equations - [click to learn more](#).
- Stress tensor is assumed to be objective,  $\mathbb{D}$  is objective

[Not a big fan of objectivity? Click here for a derivation which gets rid of the uncertainty about which rate to use: 3D analog!](#)

# Role of viscoelasticity in mantle convection models



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Geodynamical texts often use  $\sigma_{ij} = 2\eta\dot{\varepsilon}_{ij}$  for viscous rheology - that is incorrect. There are four possible strain tensors: eulerian/lagrangian, full/linearized

$$\varepsilon^{\text{lin}} := \frac{1}{2} (\nabla u + (\nabla u)^T);$$

$$\varepsilon := \frac{1}{2} (\mathbb{I} - \mathbb{F}^{-T} \mathbb{F}^{-1});$$

$$\xi^{\text{lin}} := \frac{1}{2} (\text{Grad} U + (\text{Grad} U)^T)$$

$$\xi := \frac{1}{2} (\mathbb{F}^T \mathbb{F} - \mathbb{I})$$

which have the following material time derivatives (neither one is equal to the strain rate - which is what we have in viscous rheology)

$$\dot{\varepsilon}^{\text{lin}} = \mathbb{D} - \frac{1}{2} ((\nabla u) \mathbb{L} + \mathbb{L}^T (\nabla u)^T)$$

$$\dot{\xi}^{\text{lin}} = \mathbb{D} - \frac{1}{2} ((\nabla u)^T \mathbb{L}^T + \mathbb{L} (\nabla u))$$

$$\dot{\varepsilon} = \mathbb{D} - (\varepsilon \mathbb{L} + \mathbb{L}^T \varepsilon)$$

$$\dot{\xi} = \mathbb{F}^T \mathbb{D} \mathbb{F}$$

Maxwell rheology is based on the following idea: Elastic and viscous deformations add together, but stress is the same for both elastic and viscous parts of the deformation

$$\tau = -p + \sigma; \quad \mathbb{D} = \mathbb{D}^{\text{vis}} + \mathbb{D}^{\text{el}}$$

$$\sigma = 2\eta \mathbb{D}^{\text{vis}} = 2G \varepsilon^{\text{el}}$$

The choice of  $\varepsilon$  or  $\xi$  or  $\varepsilon^{\text{lin}}$  or  $\xi^{\text{lin}}$  gives us the relation  $\varepsilon^{\text{el}} \rightarrow \mathbb{D}^{\text{el}}$  and thus specifies the rate used in our rheology (only for  $\varepsilon$  we get an objective rate - lower convected one). **In this way the stress rate is directly an outcome of the method of derivation of the constitutive equation (contrary to 1D analogs)**

# Role of viscoelasticity in mantle convection models



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$$dt^{n-1,n} = \text{stability}(\vec{v}^{n-1}, \mathcal{D}, d); \quad dt^a; \quad dt^{el}$$

$$T^n = T^{n-1} \pm \text{diff/adv} (T^{n-1}, \vec{v}^{n-1}, dt^a)$$

$$\tilde{\sigma}^{n-1} = \sigma^{n-1} - \text{adv/rot} (\sigma^{n-1}, \vec{v}^{n-1}, dt^a)$$

$$\eta^n = \eta(T^n); \quad \eta_{\text{num}}^n = \eta_{\text{num}}^n(\eta^n, dt^{el}, \mathcal{D}^n)$$

$$\text{rhs}^n = \nabla \cdot \left( \frac{\mathcal{D}^n}{dt^{el} + \mathcal{D}^n} \tilde{\sigma}^{n-1} \right) - \text{Ra} T^n \vec{e}_z$$

$$\text{Stokes}(\eta_{\text{num}}^n, \vec{v}^n, p^n) = \text{rhs}^n$$

$$\sigma^n = \sigma^n(\tilde{\sigma}^{n-1}, \eta_{\text{num}}^n, \vec{v}^n)$$

## Algorithm – time step decoupling

Algorithm that allows for different time steps in thermal and constitutive equations. Stokes is solved for every time step (whichever is smaller), which properly captures changes in the velocity field in the viscous part of the domain – contrary to the decoupling usually implemented in regional models, where velocity field is assumed not to change much within one elastic time step. You can click on the scheme to see the equations behind each step of the algorithm (i.e. the discretization we use).

Click on this text to learn more about viscoelastic and viscous parts of the domain.

# Role of viscoelasticity in mantle convection models



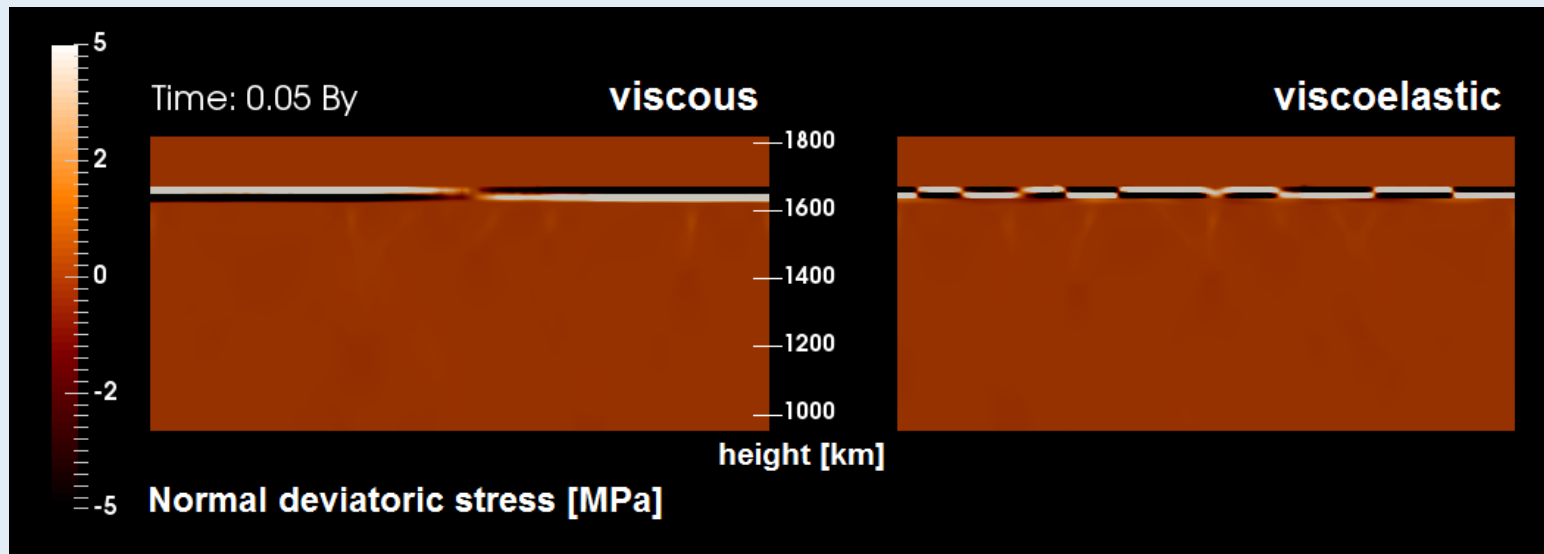
Vojtěch Patočka<sup>1,2</sup>, Ondřej Čadež<sup>1</sup>, Paul Tackley<sup>2</sup>  
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Initially thin lithosphere undergoes bending which has shorter wavelength when viscoelasticity is considered. The stress structures formed during the cooling can be remembered for times comparable to the Maxwell relaxation time of the lid.



# Role of viscoelasticity in mantle convection models



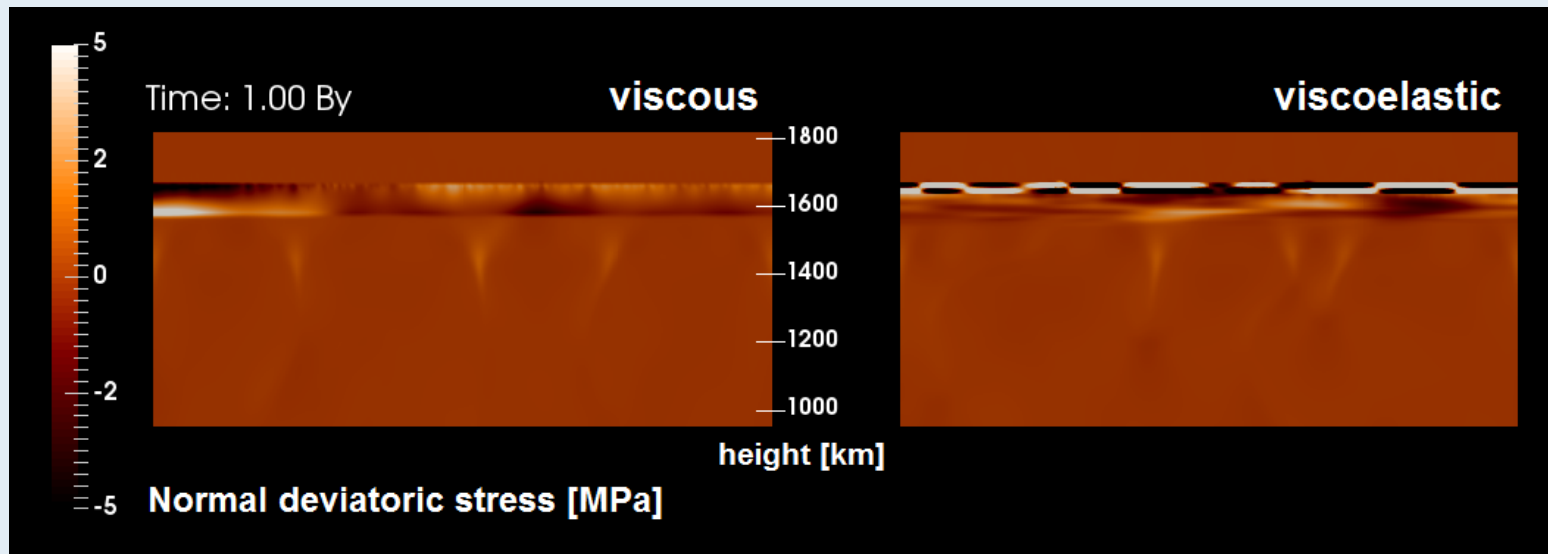
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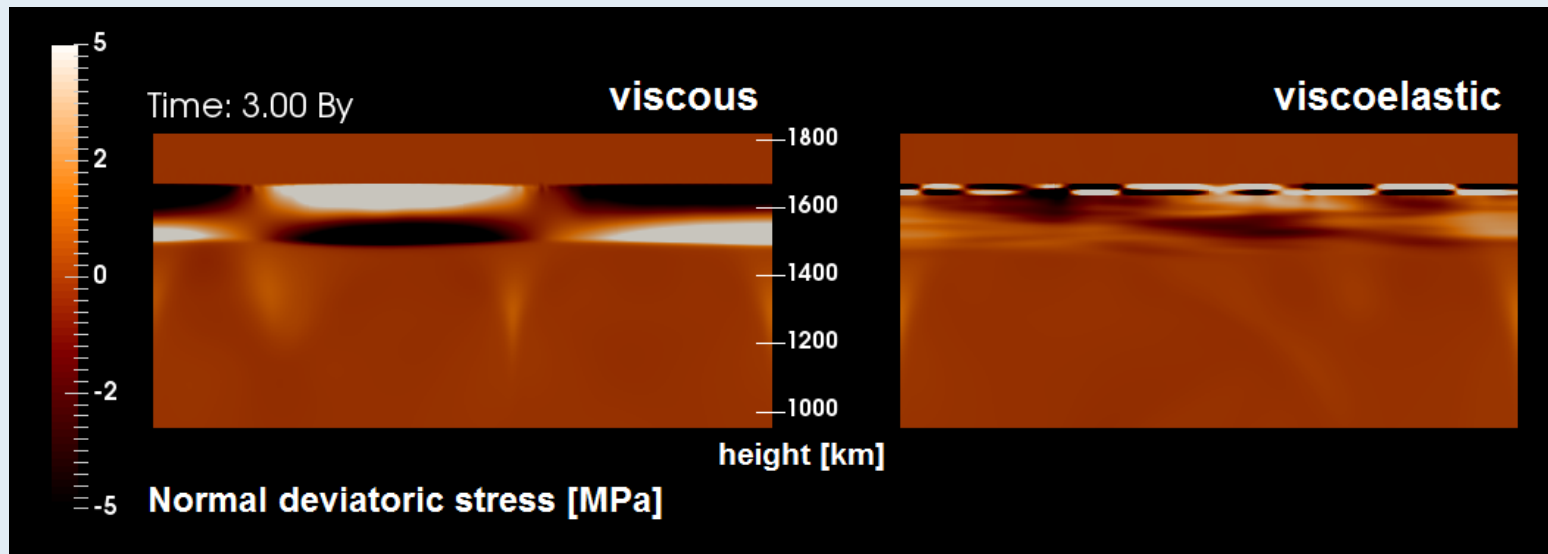
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# Role of viscoelasticity in mantle convection models



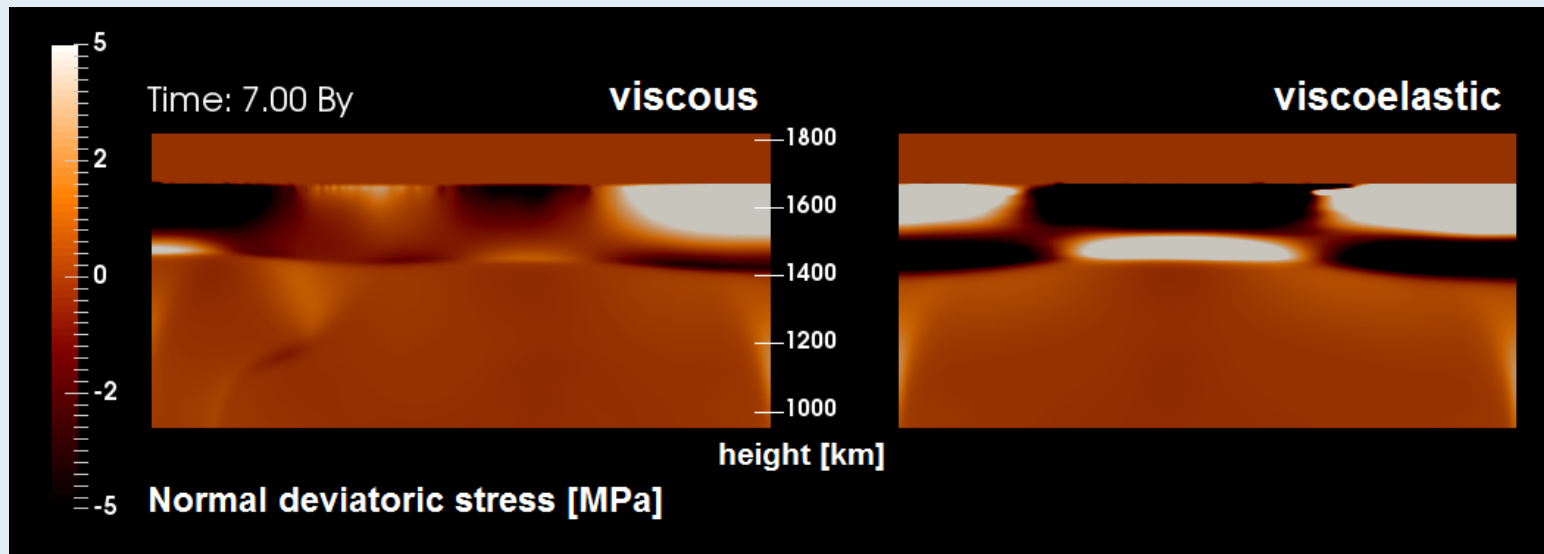
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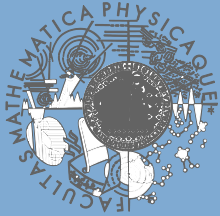


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Constitutive equation

$$\sigma + \mathcal{D}\sigma = 2\eta\mathbb{D} \quad \mathcal{D}(p, T) := \tau_{\text{rel}} \frac{\kappa}{h^2} = \frac{\eta\kappa}{Gh^2}$$

$$\overset{\nabla}{\sigma} := \frac{\partial \sigma}{\partial t} + \vec{v} \cdot \nabla \sigma + (\sigma \mathbb{W} - \mathbb{W} \sigma) - a(\mathbb{D}\sigma + \sigma \mathbb{D})$$

Discretization (mixed euler, 1st order accurate)

$$\sigma^n = \frac{dt}{dt + \mathcal{D}} 2\eta \mathbb{D}^n + \frac{\mathcal{D}}{dt + \mathcal{D}} \sigma^{n-1} + \frac{\mathcal{D} dt}{dt + \mathcal{D}} (\mathbb{W} \sigma - \sigma \mathbb{W} + a(\mathbb{D}\sigma + \sigma \mathbb{D}) - \vec{v} \cdot \nabla \sigma)$$

$$\sigma^n = \frac{dt}{dt + \mathcal{D}^n} 2\eta^n \mathbb{D}^n + \frac{\mathcal{D}^n}{dt + \mathcal{D}^n} \left( \sigma^{n-1} + dt (\mathbb{W} \sigma - \sigma \mathbb{W} + a(\mathbb{D}\sigma + \sigma \mathbb{D}) - \vec{v} \cdot \nabla \sigma)^{n-1} \right)$$

$$\tilde{\sigma}^{n-1} := \sigma^{n-1} + dt (\mathbb{W} \sigma - \sigma \mathbb{W} + a(\mathbb{D}\sigma + \sigma \mathbb{D}) - \vec{v} \cdot \nabla \sigma)^{n-1}$$

$$\sigma^n = 2\eta_{\text{num}}^n \mathbb{D}^n + \frac{\mathcal{D}^n}{dt + \mathcal{D}^n} \tilde{\sigma}^{n-1}; \quad \eta_{\text{num}} := \frac{dt}{dt + \mathcal{D}} \eta$$

Viscoelasticity parameter Z

$$Z := \frac{dt^{\text{el}}}{dt^{\text{el}} + \mathcal{D}}; \quad \sigma^n = \underbrace{2Z\eta^n \mathbb{D}^n}_{\text{bulid-up}} + \underbrace{(1-Z)\tilde{\sigma}^{n-1}}_{\text{relaxation}}$$

**Stress tensor at each time step is calculated from two parts: (1.) Stress build-up due to viscous friction and current velocity field, and (2.) stress from previous time step, which is advected and rotated to current position of the volume element and relaxed as viscoelastic relaxation takes place between time steps.**

**In a realistic mantle convection model major part of the domain is effectively purely viscous – material relaxes completely within one computational time step.**

**Click on the Viscoelasticity parameter Z to see how the elastic time step works as a filtr on elastic effects in your model.**

# Role of viscoelasticity in mantle convection models



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$$Z := \frac{dt^{el}}{dt^{el} + \mathcal{D}}; \quad \sigma^n = \underbrace{2Z\eta^n \mathbb{D}^n}_{\text{bulid-up}} + \underbrace{(1-Z)\tilde{\sigma}^{n-1}}_{\text{relaxation}}$$

Back to algorithm

Let's take a look at the value of viscoelasticity parameter  $Z$ . If elastic time step is set to 0.01 of the reference relaxation time (i.e. the relaxation time of material with reference viscosity and shear modulus), then  $Z$  has the following value:

$$\mathcal{D} = \eta \mathcal{D}_0; \quad dt^{el} = 0.01 \tau_{rel} \frac{\kappa}{h^2} = 0.01 \mathcal{D}_0; \quad Z = \frac{1}{1 + \frac{\eta}{0.01}};$$

In that part of the domain where viscosity is much smaller than 0.01 of the reference viscosity  $Z$  will be close to 1 and thus purely visous behavior will be observed. Elastic time step can be considered a filter on the elastic effects in the model: Elastic effects will be observed only where relaxation time is not smaller than computational time step.

This causes numerical problems: If there is no lower threshold on the elastic time step, the elastic part of the domain grows as time step is getting smaller – this prohibits the viscous part of the domain to stop elastic instantaneous deformation. By decreasing the time step we are effectively increasing the effect of elasticity in the model.

# Role of viscoelasticity in mantle convection models



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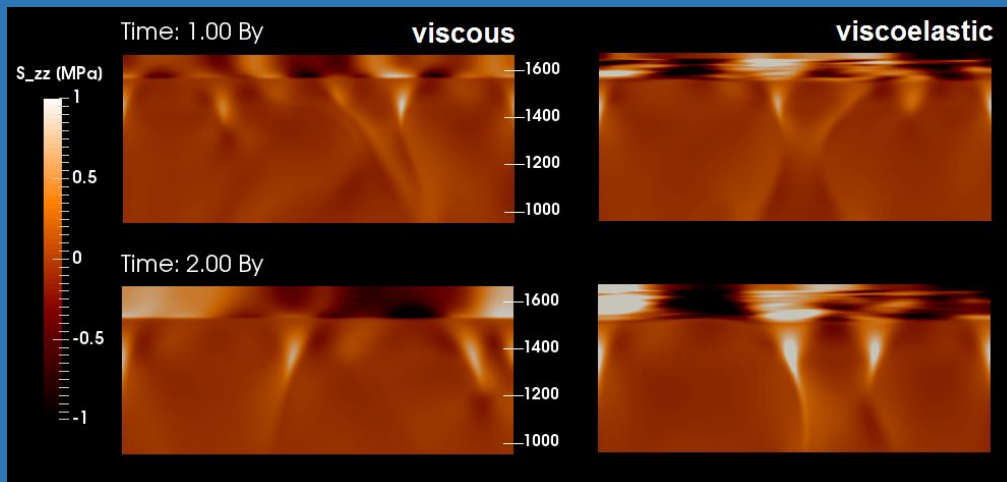
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## Results – the memory effect

The distribution of stresses in the lithosphere differs significantly when viscoelasticity is considered. This can influence processes which take place in the lithosphere, or even the regime of convection. Especially in the early stages of planetary evolution.

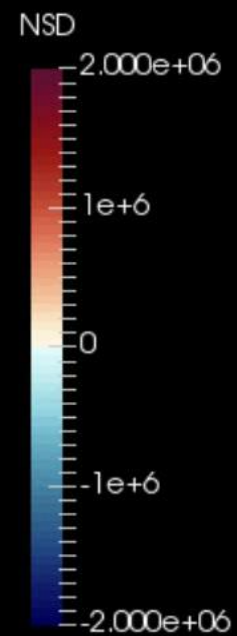
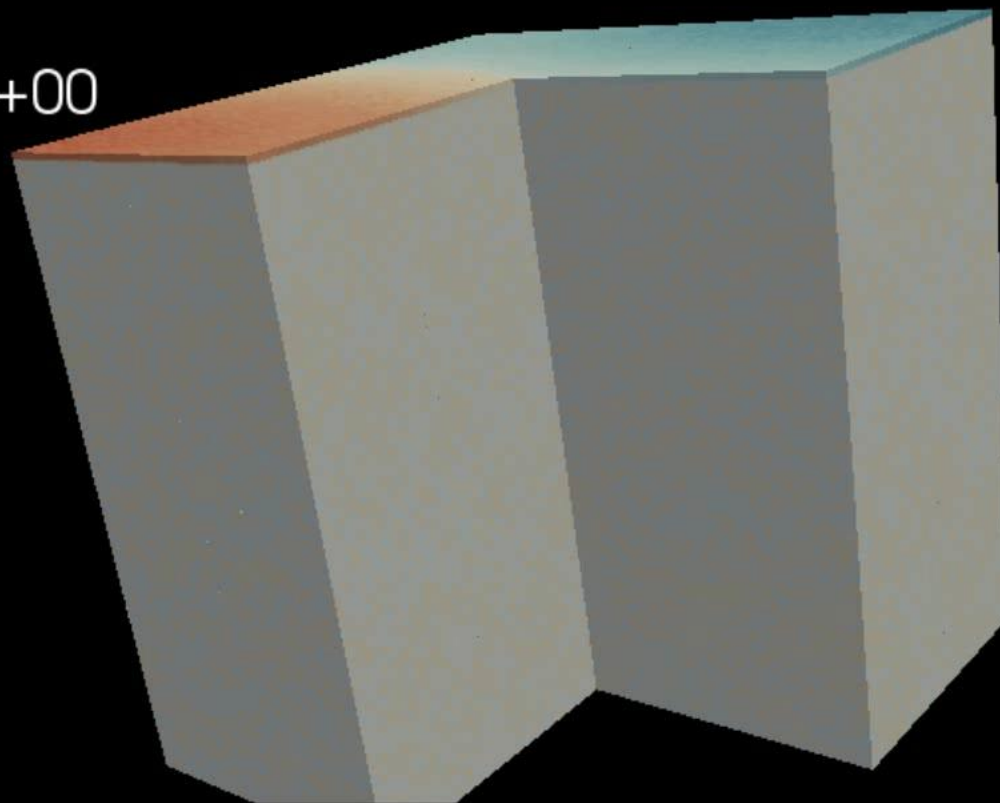


Viscoelastic models remember the deformation which is taking place in the lithosphere during its formation. The figure shows a simulation with a free-slip surface. Click on it to see one with a free surface.

Just a movie, please:

**3D**

Time: 0.00e+00



# Role of viscoelasticity in mantle convection models



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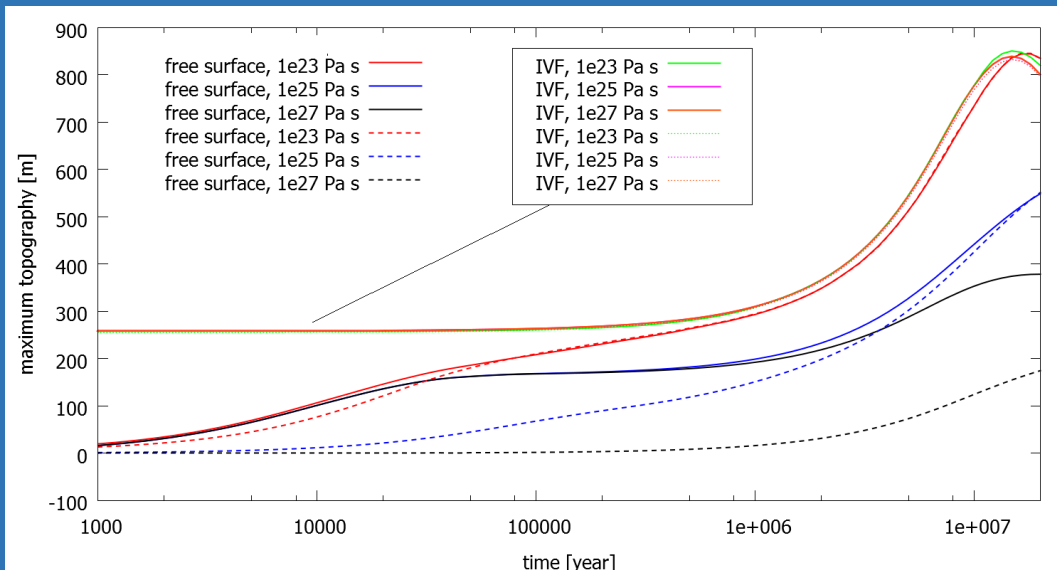


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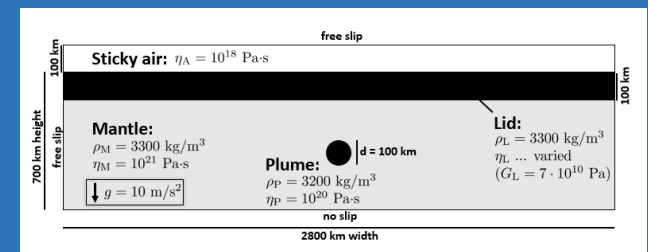
## Results – topography above a rising cylinder

Viscoelasticity reduces the discrepancy between free surface and traction derived topographies, which occurs for high viscosity lids. Here we results from a test case by Crameri et al., 2012, extended by including high viscosity, viscoelastic lids.

IVF stands for traction derived topographies.



Click on the picture to enlarge the model setup



# Role of viscoelasticity in mantle convection models



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## Results – topography above a rising cylinder

Model setup for a test case designed by Crameri et al., 2012, extended to study high viscosity, viscoelastic lids.

