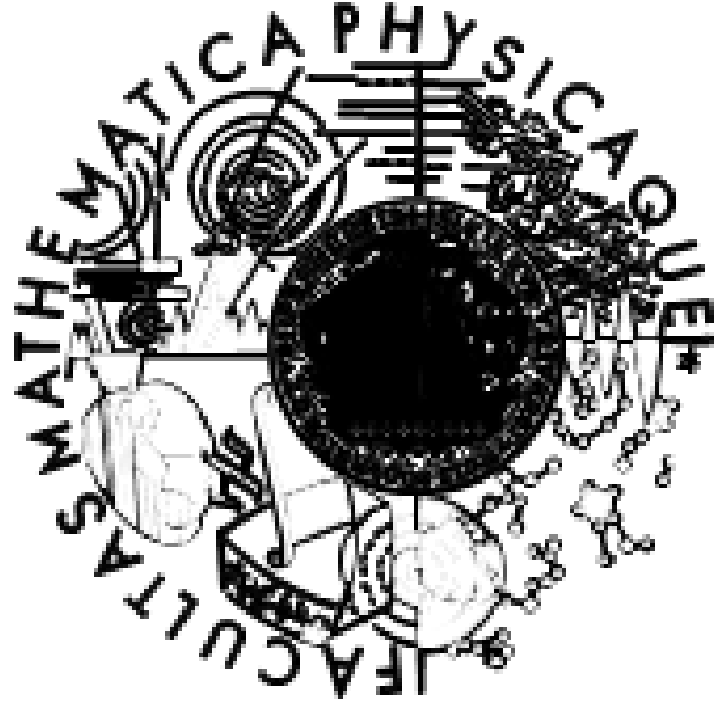


Role of viscoelasticity in mantle convection models



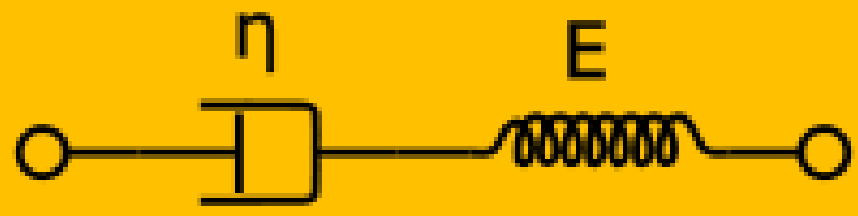
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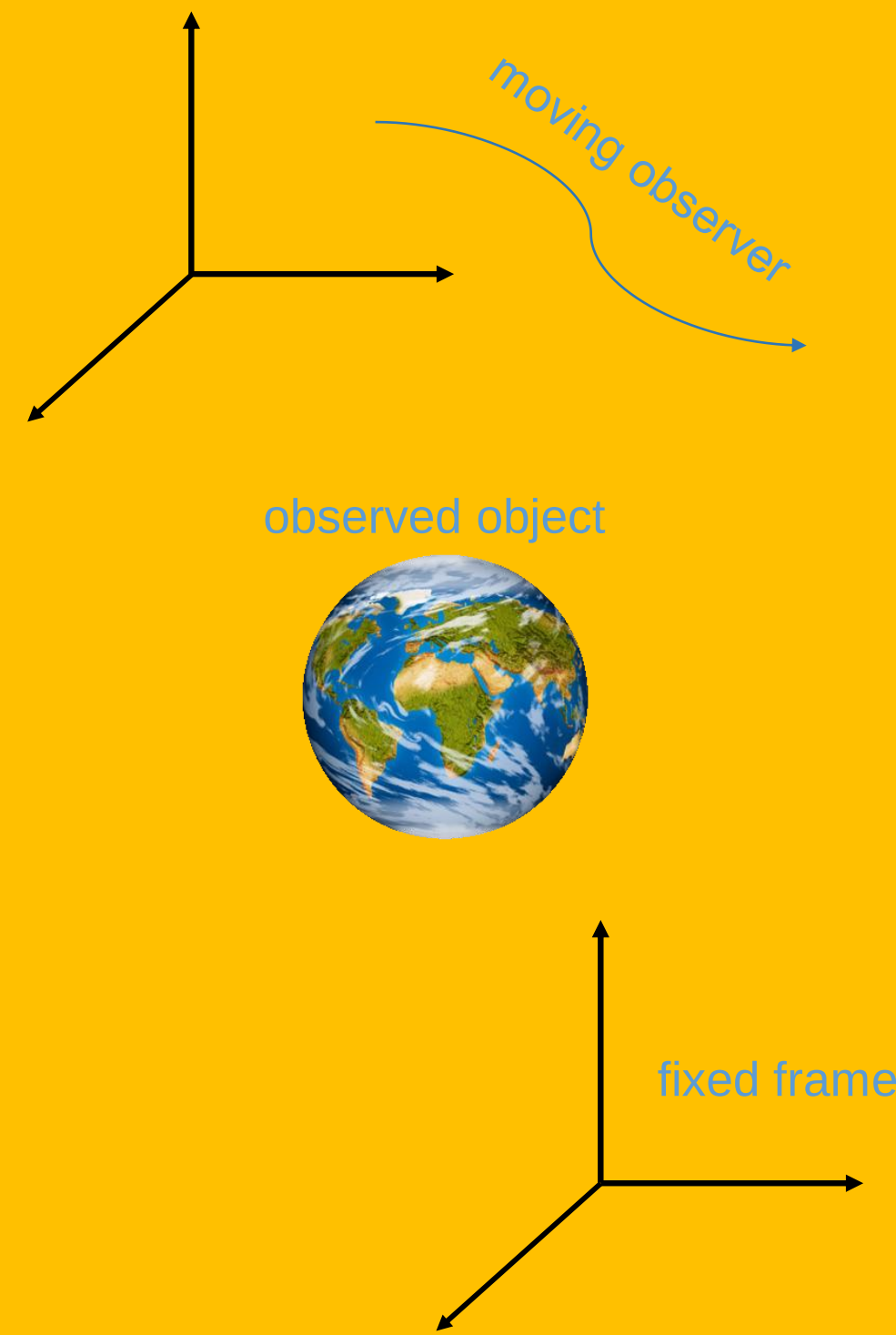


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1D Analog – Linear viscoelasticity



- Maxwell element - spring and dashpot in series
- Forces equal: $F = F_S = F_D$, Deformation adds up: $\Delta = \Delta_S + \Delta_D$
- Constitutive relation: $\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = 2\mathbb{D}$, $\mathbb{D} \dots$ Strain rate
- Physical equations must be written in terms of geometrical objects - i.e. scalars and co/contra-variant tensors of first or higher order.
- Objectivity - material time derivative of a tensor is geometrically not a tensor! ($\dot{\mathbb{A}}^* \neq \mathbb{Q}\dot{\mathbb{A}}\mathbb{Q}^T$)
- Only special (objective) stress rate may thus be chosen in the constitutive law



3D Analog – Stress rate obtained as a result

Geodynamical texts often use $\sigma_{ij} = 2\eta\dot{\epsilon}_{ij}$ for viscous rheology - that is incorrect. There are four possible strain tensors: eulerian/lagrangian, full/linearized

$$\begin{aligned}\dot{\epsilon}^{\text{lin}} &:= \frac{1}{2}(\nabla u + (\nabla u)^T); & \dot{\xi}^{\text{lin}} &:= \frac{1}{2}(\text{Grad} U + (\text{Grad} U)^T) \\ \dot{\epsilon} &:= \frac{1}{2}(\mathbb{I} - \mathbb{F}^{-T}\mathbb{F}^{-1}); & \dot{\xi} &:= \frac{1}{2}(\mathbb{F}^T\mathbb{F} - \mathbb{I})\end{aligned}$$

which have the following material time derivatives (neither one is equal to the strain rate - which is what originally figures in the definition of viscous rheology)

$$\begin{aligned}\dot{\epsilon}^{\text{lin}} &= \mathbb{D} - \frac{1}{2}((\nabla u)\mathbb{L} + \mathbb{L}^T(\nabla u)^T) & \dot{\xi} &= \mathbb{D} - (\dot{\epsilon}\mathbb{L} + \mathbb{L}^T\dot{\epsilon}) \\ \dot{\xi}^{\text{lin}} &= \mathbb{D} - \frac{1}{2}((\nabla u)^T\mathbb{L}^T + \mathbb{L}(\nabla u)) & \dot{\xi} &= \mathbb{F}^T\mathbb{D}\mathbb{F}\end{aligned}$$

Maxwell rheology is based on the following idea: Elastic and viscous deformations add together, but stress is the same for both elastic and viscous parts of the deformation

$$\tau = -p + \sigma; \quad \mathbb{D} = \mathbb{D}^{\text{vis}} + \mathbb{D}^{\text{el}}$$

$$\sigma = 2\eta\mathbb{D}^{\text{vis}} = 2G\dot{\epsilon}^{\text{el}}$$

The choice of $\dot{\epsilon}$ or $\dot{\xi}$ or $\dot{\epsilon}^{\text{lin}}$ or $\dot{\xi}^{\text{lin}}$ gives us the relation $\dot{\epsilon}^{\text{el}} \rightarrow \mathbb{D}^{\text{el}}$ and thus specifies the rate used in our rheology (only for $\dot{\epsilon}$ we get an objective rate - lower convected one). In this way the stress rate is directly an outcome of the method of derivation of the constitutive equation (contrary to 1D analogs)

1D analog



Objectivity

(fundamental constraint)

or

3D analog

(Stress rate specified automatically)

VISCOELASTIC RHEOLOGY
 $\sigma + \mathcal{D}\dot{\sigma} = 2\eta\mathbb{D}$

$$\sigma + \mathcal{D}\dot{\sigma} = 2\eta\mathbb{D} \quad \mathcal{D}(p, T) := \tau_{\text{rel}} \frac{\kappa}{h^2} = \frac{\eta\kappa}{Gh^2}$$

$$\dot{\sigma} := \frac{\partial \sigma}{\partial t} + \vec{v} \cdot \nabla \sigma + (\sigma \mathbb{W} - \mathbb{W} \sigma) - \mathbf{a}(\mathbb{D} \sigma + \sigma \mathbb{D})$$

advection co-rotation co-deformation

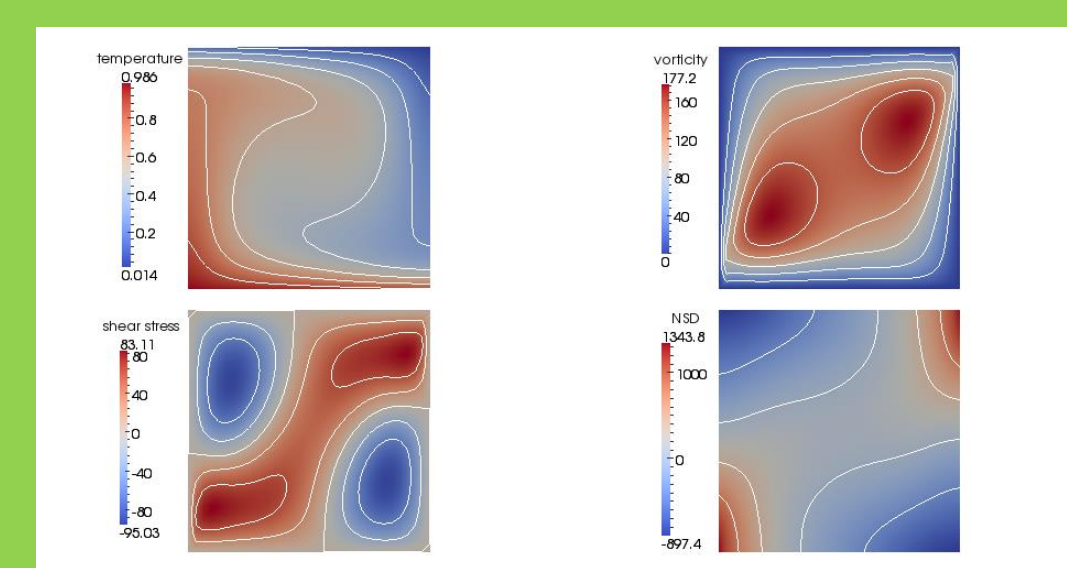
$\sigma \dots$ Stress, $\mathcal{D} \dots$ Deborah number, $\mathbb{D} \dots$ Strain rate

Implementation algorithm:
decoupling vs.
no decoupling

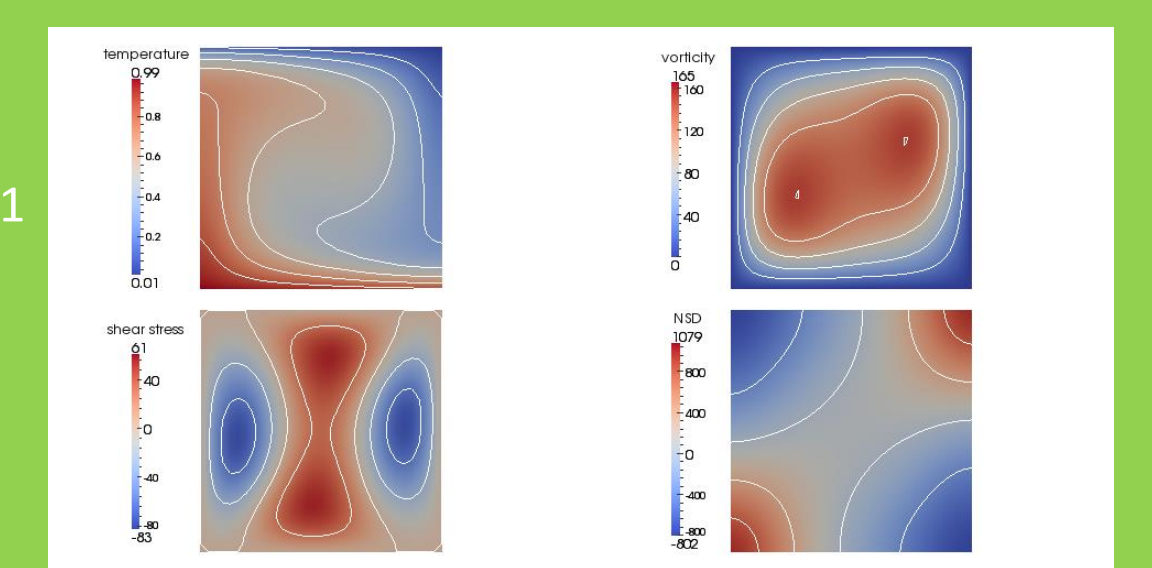
Numerical instability:
Instantaneous
deformation

Results and goals:
Stress distribution in
the lithosphere

We implement maxwellian viscoelastic rheology into StagYY in order to study how changes in the stress distribution in the lithosphere influence tectonic processes on terrestrial planets. To do so, one needs to tackle a general problem: viscoelastic material responds with instantaneous deformation when forces applied to the material undergo instantaneous change. This results in a critical Deborah number above which simulations experience singularity in the velocity field.



Co-rotational stress rate



Viscous

Harder et al. 1991
can be used for
testing the
implementation

Algorithm – time step decoupling

```
dtn-1,n = stability( $\tilde{v}^{n-1}$ ,  $\mathcal{D}$ ,  $d$ );  $dt^a$ ;  $dt^{el}$ 

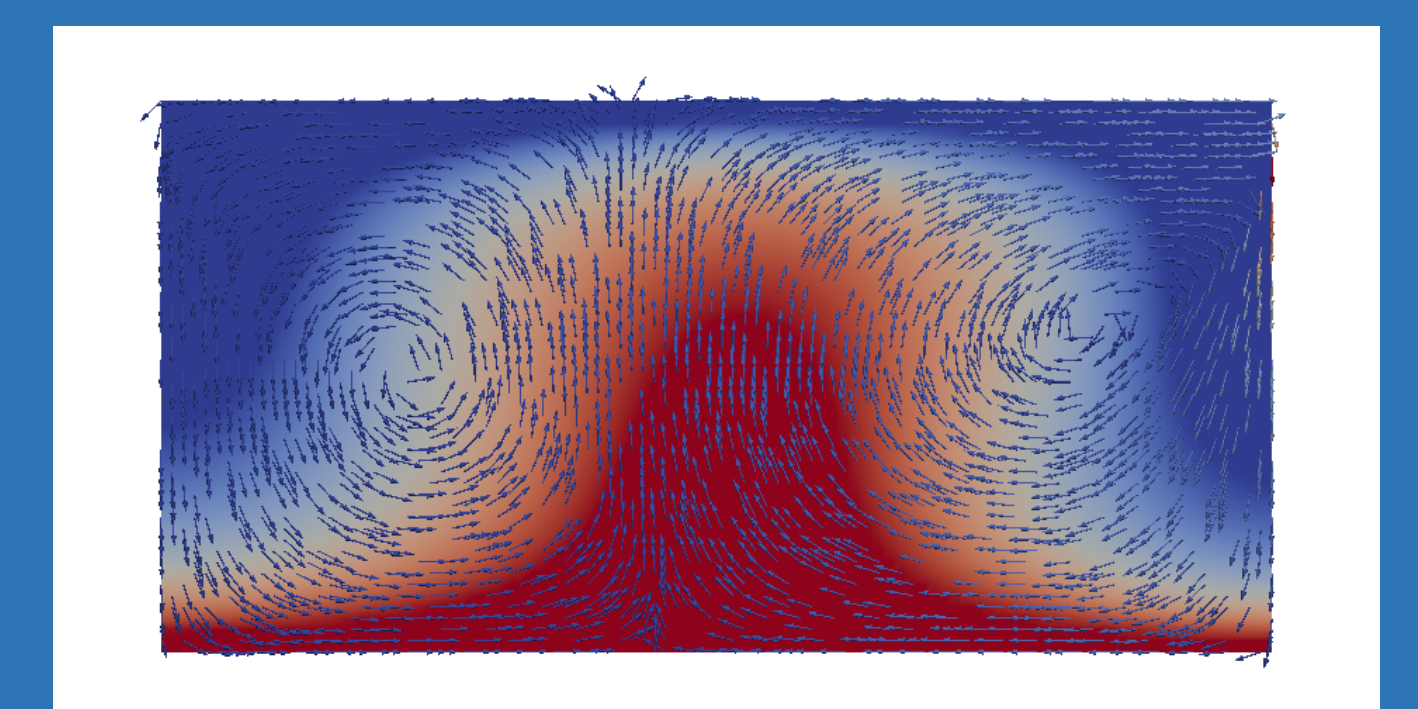
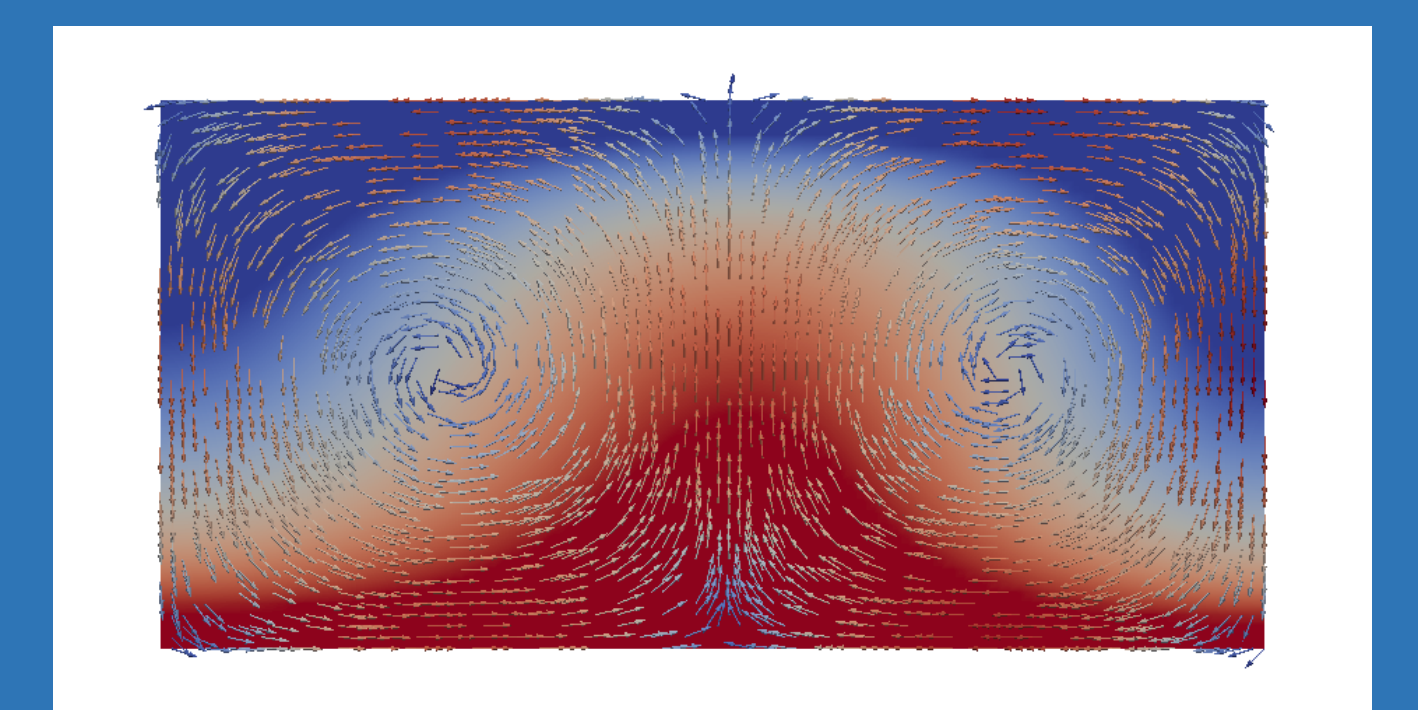
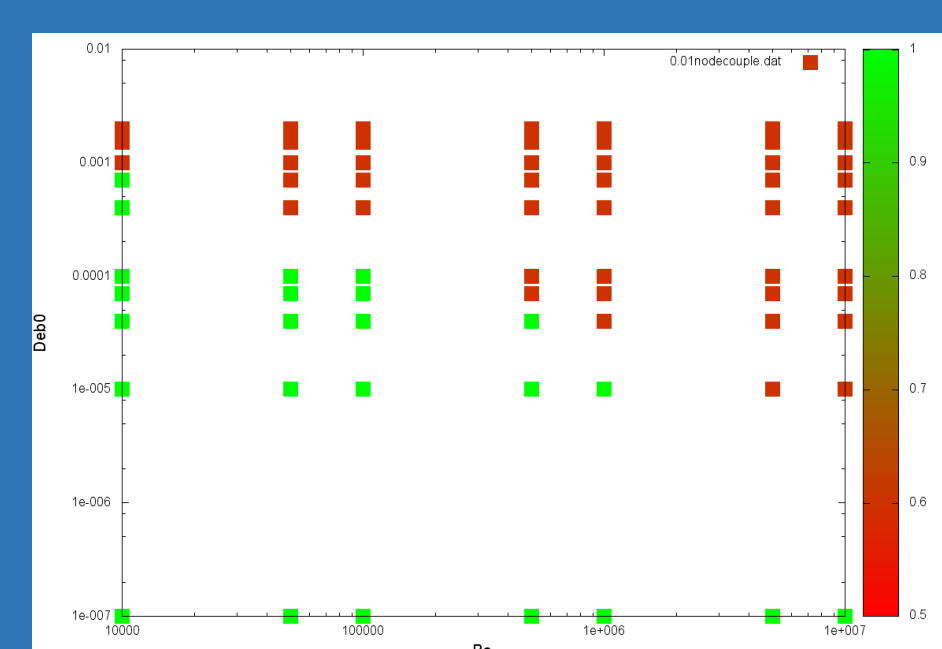
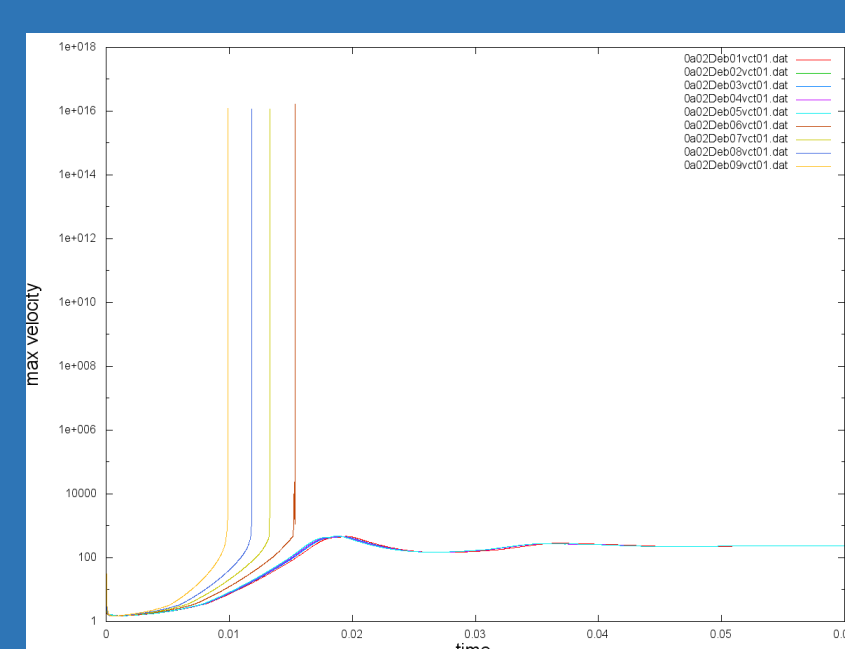
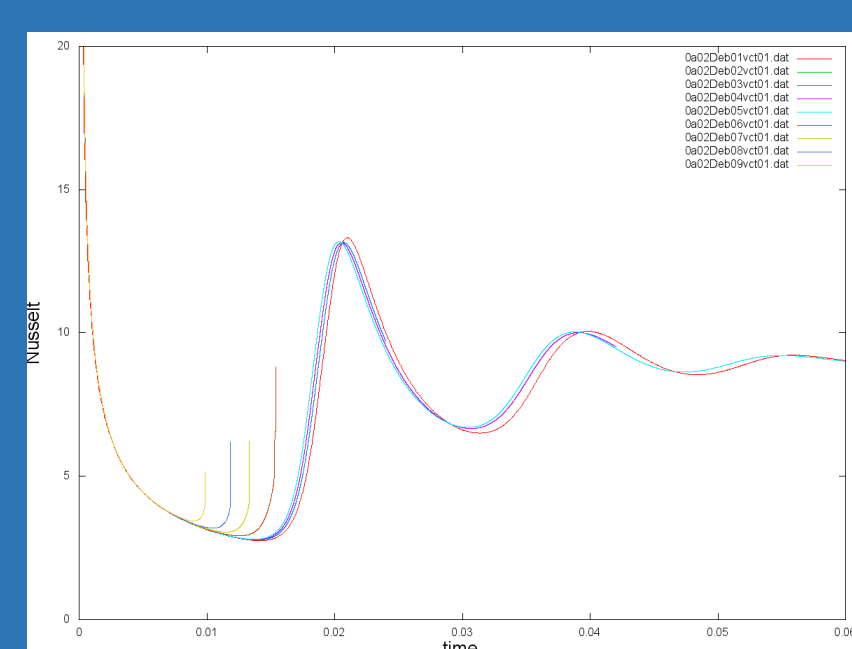
 $T^n = T^{n-1} \pm \text{diff/adv}(T^{n-1}, \tilde{v}^{n-1}, dt^a)$ 
 $\tilde{\sigma}^{n-1} = \sigma^{n-1} - \text{adv/rot}(\sigma^{n-1}, \tilde{v}^{n-1}, dt^a)$ 
 $\eta^n = \eta(T^n); \quad \eta_{\text{num}}^n = \eta_{\text{num}}(\eta^n, dt^{el}, \mathcal{D}^n)$ 

 $\text{rhs}^n = \nabla \cdot \left( \frac{\mathcal{D}^n}{dt^{el} + \mathcal{D}^n} \tilde{\sigma}^{n-1} \right) - \text{Ra} T^n \vec{e}_z$ 

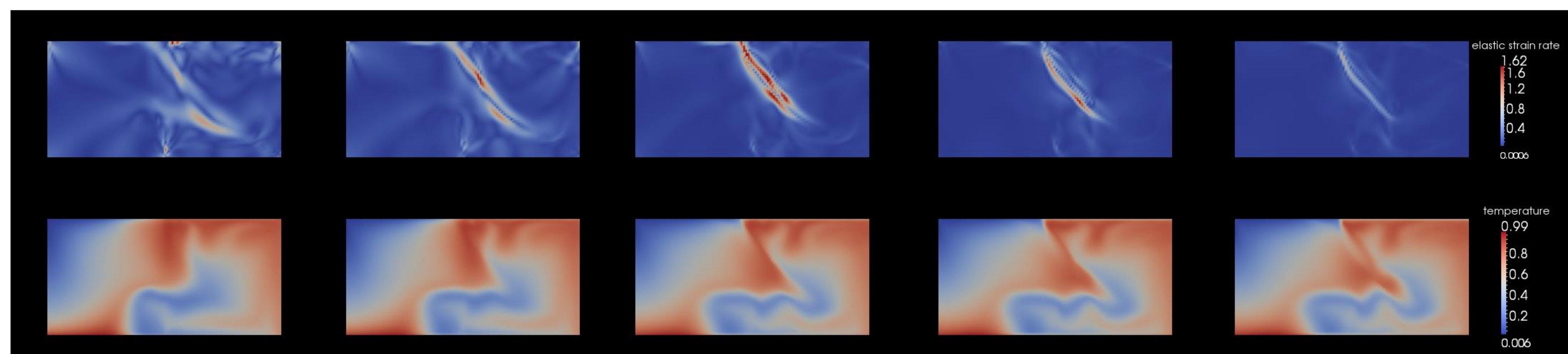
 $\text{Stokes}(\eta_{\text{num}}^n, \tilde{v}^n, p^n) = \text{rhs}^n$ 
 $\sigma^n = \sigma^n(\tilde{\sigma}^{n-1}, \eta_{\text{num}}^n, \tilde{v}^n)$ 
```

Numerical instability – effectively instantaneous deformation

Onset of convection is related to “numerically instantenous” elastic behavior for Deborah number above certain critical value. This causes a problem since velocities in that part of the domain go to infinity and thus CFL condition stops the simulation. One can set a minimum value for the time step used in the discretized constitutive equation to damp this instability. Locally large velocities then influence only time step used in the thermal equation.



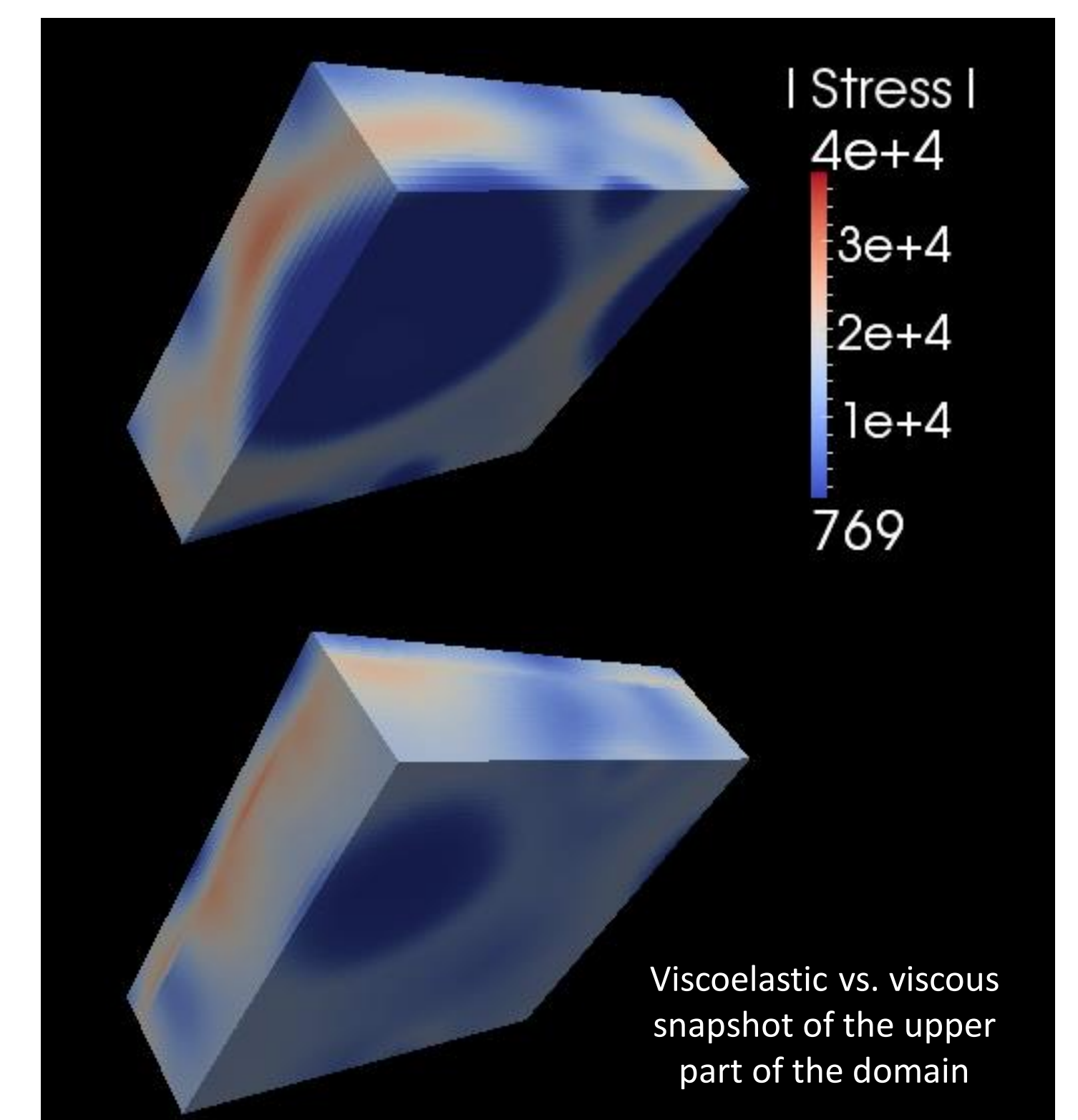
Results – elastic epizodes:



Future directions:

Implementation of visco-elasto-plasticity (with variable shear modulus) into 3D spherical StagYY will follow. We look for suitable benchmarks to test the implementation of elasticity when doing so. Elastic events have to be parametrized to avoid obtaining results with velocities which are outside the scope of infinite Prandtl number convection.

Results – stress in stagnant lid:



Viscoelastic vs. viscous
snapshot of the upper
part of the domain