

Polar wander of terrestrial bodies: solving the non-linear Liouville equation

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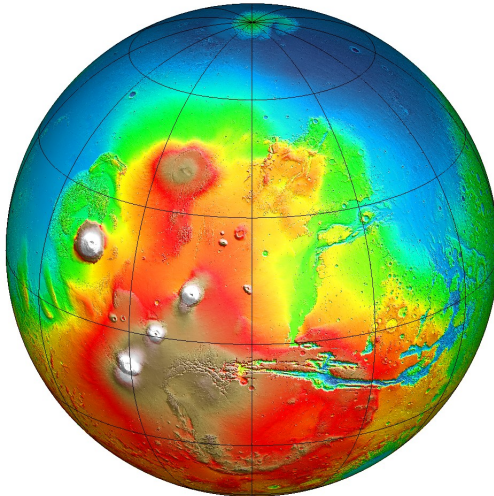
<http://geo.mff.cuni.cz/users/patocka/>

15th February 2019

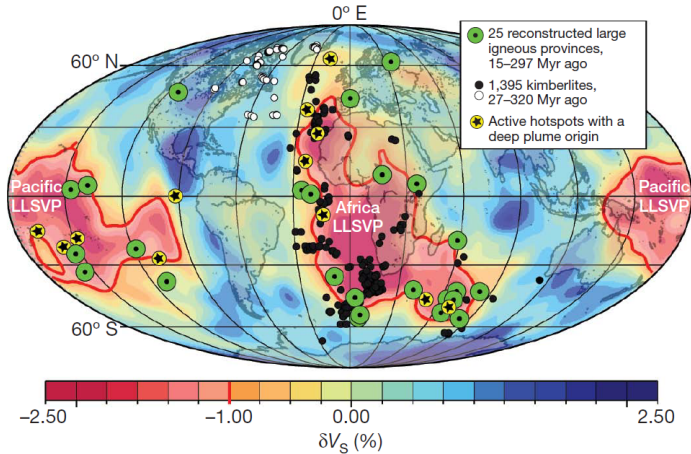
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 - Stationary solution
- 2 The Earth: postglacial rebound
 - Viscoelastic response of the mantle
- 3 Energy balance
 - Gravitational energy
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 - Forces $\leftrightarrow$ Energies

Tharsis on Mars

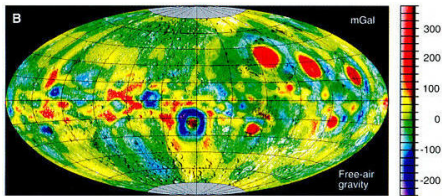


Topographic maps produced by the Mars Orbiter Laser Altimeter



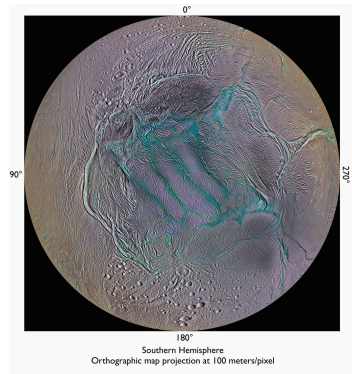
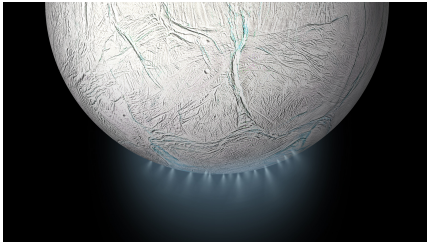
Torsvik et al., 2010: Diamonds sampled by plumes from CMB, Nature

Mascons on the Moon



Free air gravity anomaly map of the Moon (from Zuber et al., 1994),
distribution of impact basins on the trailing hemisphere (Nasa)

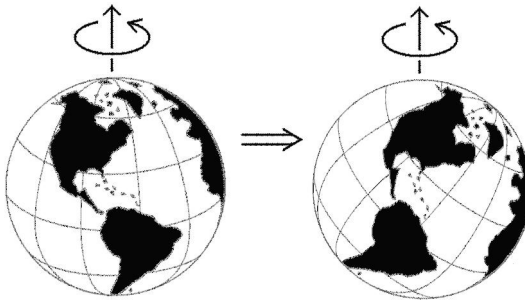
Enceladus (credits: NASA/JPL-Caltech)



The location of Tiger stripes on Enceladus: tidal dissipation, or polar wander?

How does polar wander work?

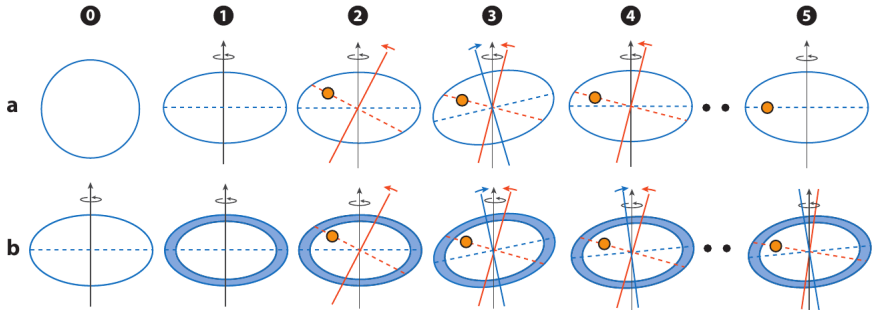
$$0 = \frac{d}{dt}(\mathbf{I} \cdot \boldsymbol{\omega}) + \boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega}), \quad \text{Stationary : } \boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega}) = 0$$



Note the difference: true polar wander vs apparent polar wander

Reorienting a planet, the remnant bulge

Without remnant bulge, any load fully reorients the planet. With a remnant bulge, the stationary solution balances the load contribution over the remnant bulge.



MacCullagh's formula

$$I_{ij} = I_0 \delta_{ij} + MR^2 \begin{pmatrix} \frac{1}{3}C_{20} - 2C_{22} & -2S_{22} & -C_{21} \\ -2S_{22} & \frac{1}{3}C_{20} + 2C_{22} & -S_{21} \\ -C_{21} & -S_{21} & -\frac{2}{3}C_{20} \end{pmatrix}.$$

axially symmetric loading:

$$I_{ij} = I_0 \delta_{ij} + MR^2 C'_{20} \left(\frac{1}{3} \delta_{ij} - e_i e_j \right)$$

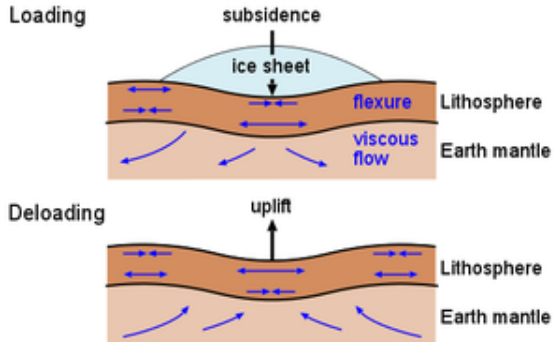
Mass load

Rotational deformation

Remnant bulge

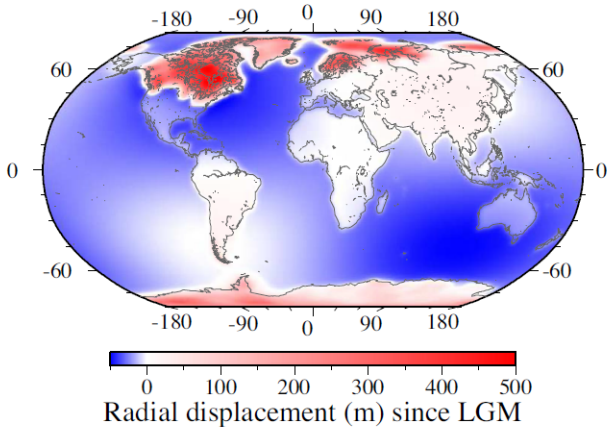
$$C_{20}^{LD'} = \frac{1 + k_2^L}{5} \sigma'_{20}; \quad C_{20}^{RD'} = -k_2^T \frac{\Omega^2 R^3}{3GM}; \quad C_{20}^{RD'} = (k_2^T - k_2^{T*}) \frac{\Omega^{*2} R^3}{3GM}$$

Postglacial rebound



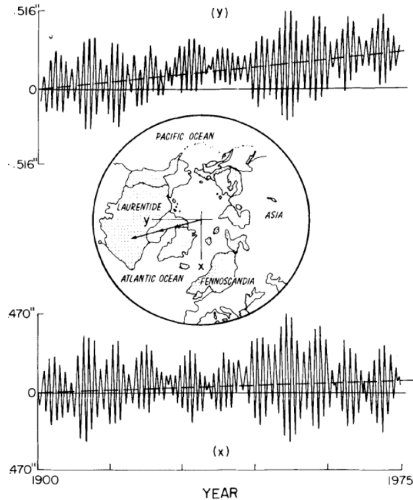
The stationary solution does not predict any polar wander: the load appears and then disappears, leaving the distribution of the body's mass unaltered

Postglacial rebound



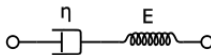
from Martinec and Hagedoorn, 2014, GJI

Postglacial rebound



Wu and Peltier, 1984: Pleistocene deglaciation and the Earth's rotation

Governing equations



$$\nabla \cdot \boldsymbol{\tau} + \mathbf{f} = 0 \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\boldsymbol{\tau}^d - \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) = -\frac{\mu}{\eta} \int_0^t \boldsymbol{\tau}^d dt' \quad (3)$$

Body force

$$\mathbf{f} = -\rho \nabla (V_0 + \Phi + \Psi) \doteq -(\mathbf{u} \cdot \nabla \rho_0) \mathbf{g}_0 - \rho_0 \nabla (\Phi + \Psi) + \rho_0 \mathbf{g}_0$$

Eulerian formulation

$$\begin{aligned} \delta \rho(\mathbf{r}, t) &= \rho(\mathbf{r}, t) - \rho_0(r) = \rho_0(|\mathbf{r} - \mathbf{u}(\mathbf{r}, t)|, t) - \rho_0(r) = \\ &= -\mathbf{u}(\mathbf{r}, t) \cdot \nabla \rho_0(r) + O(|\mathbf{u}|^2) \end{aligned}$$

Boundary conditions (domain: spherical shell)

$$\boldsymbol{\tau} \cdot \mathbf{e}_r = (u_r[\rho_0]_a + \sigma^L) \mathbf{g}_0 \quad (4)$$

$$-\boldsymbol{\tau} \cdot \mathbf{e}_r = u_r[\rho_0]_b \mathbf{g}_0 - \rho_c (V_0 + \Phi + \Psi) \mathbf{e}_r \quad (5)$$

Selfgravity potential Φ computed using the condensation method:

$$-\Phi_{\ell m}^{e,topo}(r) = \frac{4\pi Gr}{2\ell + 1} [\rho_0]_b u_{r,\ell m}^{CMB} \left(\frac{r_b}{r}\right)^{j+2}$$

Condensation: treating topographic load as a load with 2D surface mass-density.

Properties of the governing equations

- Unknowns: $\mathbf{u}, \boldsymbol{\tau}$
- Driving forces: $\Psi(\boldsymbol{\omega}), \sigma_L$
- Linearity in Ψ, σ_L
- Decoupling in terms of spherical harmonics

$$u_r(r, \vartheta, \varphi) = \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} u_{r,\ell m}(r) Y_{\ell m}(\vartheta, \varphi)$$

$$\Phi(r, \vartheta, \varphi) = \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} \Phi_{\ell m}(r) Y_{\ell m}(\vartheta, \varphi)$$

Our method

$$\mathbf{I} = I_0 \mathcal{I} + \sqrt{\frac{5}{\pi}} \frac{a^3}{6G} \left[\begin{pmatrix} -\Phi_{20} & 0 & 0 \\ 0 & -\Phi_{20} & 0 \\ 0 & 0 & 2\Phi_{20} \end{pmatrix} - \sqrt{6} \begin{pmatrix} -\operatorname{Re} \Phi_{22} & \operatorname{Im} \Phi_{22} & \operatorname{Re} \Phi_{21} \\ \operatorname{Im} \Phi_{22} & \operatorname{Re} \Phi_{22} & -\operatorname{Im} \Phi_{21} \\ \operatorname{Re} \Phi_{21} & -\operatorname{Im} \Phi_{21} & 0 \end{pmatrix} \right]$$

Φ is obtained by solving the small viscoelastic deformation of the mantle with. The solution is coupled to the full form of the Liouville equation

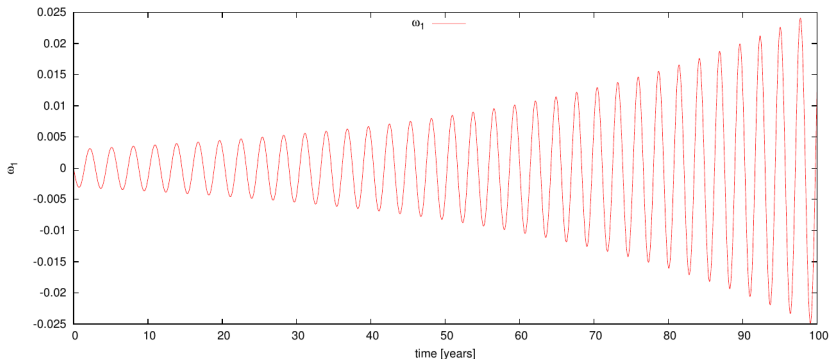
$$-\mathbf{I} \cdot \frac{d\boldsymbol{\omega}}{dt} = \frac{d\mathbf{I}}{dt} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega}) \quad \Rightarrow \quad \frac{d\boldsymbol{\omega}}{dt} = \mathbf{f}(\mathbf{I}(\boldsymbol{\omega}), \boldsymbol{\omega})$$

A set of ordinary differential equations, each time step not very expensive.

Response to (sudden) loading: free wobble

The Liouville equation responds with free (Chandler) wobble and some secular trend when a planet is subject to loading

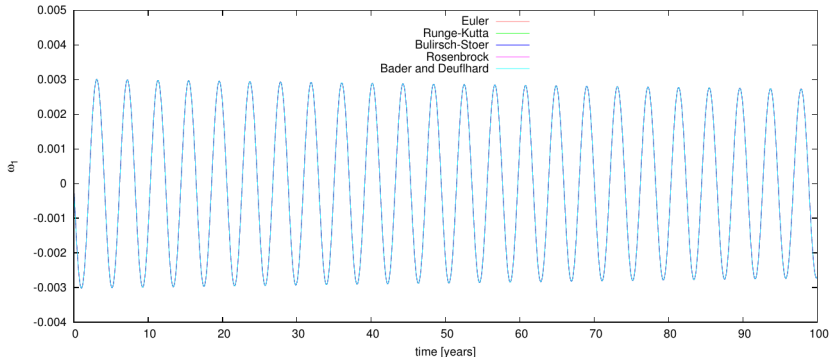
Figure 3.2: Divergence of $\omega_1(t)$ for large stepsize



Response to (sudden) loading: free wobble

The free wobble gets damped after some period of time

Figure 3.3: $m_1(t)$ computed by different methods with high precision



Involved energies

The best way to control numerical solution is by looking at the conservation of energy

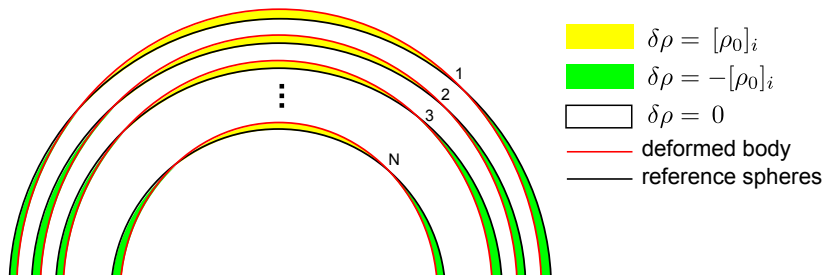
$$E_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \quad (6)$$

$$E_{\text{el}} = \int_{v(t)} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{4\mu} dv \doteq \int_{S_a-S_b} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{4\mu} dv \quad (7)$$

$$\dot{E}_{\text{diss}} = \int_{v(t)} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{2\eta} dv \doteq \int_{S_a-S_b} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{2\eta} dv \quad (8)$$

$$E_{\text{grav}} = \frac{1}{2} \int_{v(t)} \rho (V_0 + \Phi) dv \quad (9)$$

N-layered deformed body



The deformation due to loading and rotation is small: rotational bulge $\sim 10\text{km}$, radius $\sim 1000\text{km}$

Condensation method?

$$E_{\text{grav}} = \frac{1}{2} \int_{v(t)} \rho (V_0 + \Phi) dv \quad (10)$$

$$= \frac{1}{2} \int_S \rho (V_0 + \Phi) dv + \frac{1}{2} \int_h \rho (V_0 + \Phi) dv \quad (11)$$

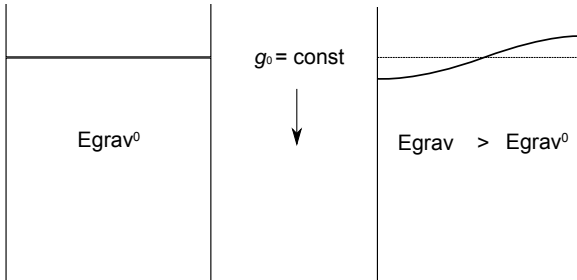
$$= \frac{1}{2} \int_S \rho V_0 dv + \frac{1}{2} \int_S \rho \Phi dv + \frac{1}{2} \int_h \rho V_0 dv + \frac{1}{2} \int_h \rho \Phi dv \quad (12)$$

The 1st term is E_{grav}^0 , 2nd and 3rd terms are killed by **condensation**, 4th term becomes:

$$E_{\text{grav}} - E_{\text{grav}}^0 = \frac{1}{2} \int_h \rho \Phi dv = \frac{1}{2} \int_{\partial S} \rho \Phi u_r ds \quad (13)$$

Gravitational energy in the referential potential

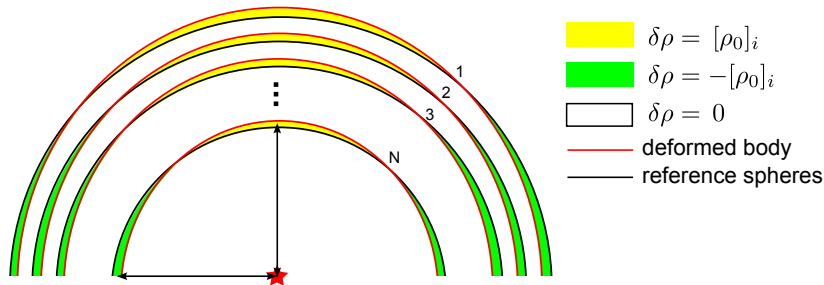
$$\frac{1}{2} \int_h \rho V_0 dv$$



Formula for the gravitational energy of non-zero topography in the referential gravitational potential

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1,N} [\rho_0]_i \int_{h_i} V_0(r) \, dv &\doteq \frac{1}{2} \sum_{i=1,N} [\rho_0]_i \int_{h_i} (V_0(r_i) + g_0(r_i)(r - r_i)) \, dv \\
 &= \frac{1}{2} \sum_{i=1,N} [\rho_0]_i g_0(r_i) \int_{h_i} r \, dv \doteq \frac{1}{2} \sum_{i=1,N} [\rho_0]_i g_0(r_i) \int_{\partial S_i} \int_{r_i}^{r_i + u_r(\mathbf{r}_i)} r \, dr \, ds \\
 &= \frac{1}{4} \sum_{i=1,N} [\rho_0]_i g_0(r_i) \int_{\partial S_i} u_r^2(\mathbf{r}_i) \, ds
 \end{aligned}$$

$$\Phi_{00} = 0 ? \quad \frac{1}{2} \int_S \rho \Phi dv = 0 ?$$



$$\frac{1}{2} \int_S \rho \Phi dv = \frac{1}{2} \int_h \rho V_0 dv$$

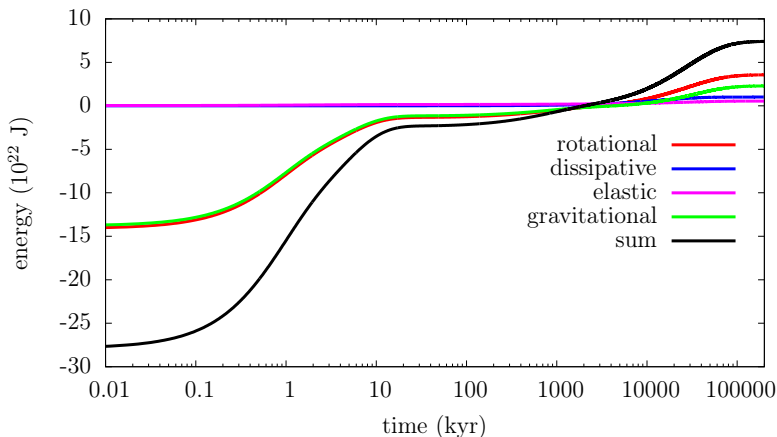
Final formula is thus (for a single layer body):

$$E_{\text{grav}} - E_{\text{grav}}^0 \doteq \frac{1}{2} \int_{\partial S} \rho (g_0 u_r^2 + \Phi u_r) \, ds, \quad (14)$$

instead of the incorrect (“condensated”):

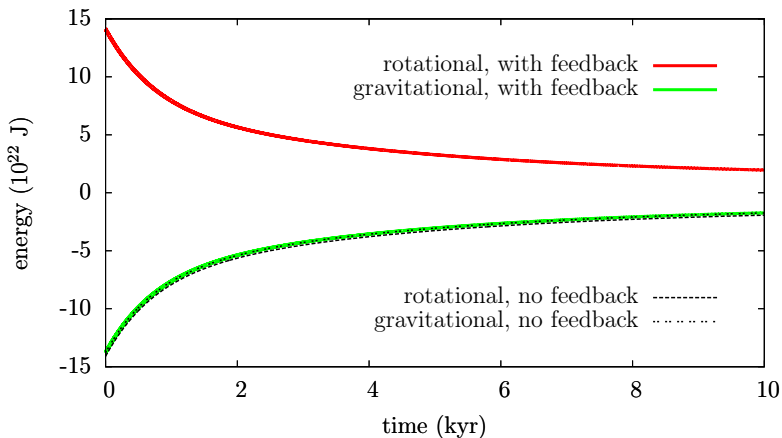
$$E_{\text{grav}} - E_{\text{grav}}^0 \doteq \frac{1}{2} \int_{\partial S} \rho \Phi u_r \, ds \quad (15)$$

Energy balance on a constantly rotating sphere



from Patočka, Čadek, and Martinec, 2018, Geophys. J. Int.

Energy balance with the rotational feedback



from Patočka, Čadek, and Martinec, 2018, Geophys. J. Int.

Powers ↔ Energies

$$\nabla \cdot \boldsymbol{\tau} - (\mathbf{u} \cdot \nabla \rho_0) \mathbf{g}_0 - \rho_0 \nabla (\Phi + \Psi) = 0 \quad (16)$$

Rates of energies

$$\frac{d}{dt} \left(\int_{S_a-S_b} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{4\mu} dv \right) + \int_{S_a-S_b} \frac{\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}{2\eta} dv$$

$$+ \frac{d}{dt} \left(\frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \right)$$

$$+ \frac{d}{dt} \left(\frac{1}{2} [\rho_0]_a \int_{\partial S_a} g_0 u_r^2 ds + \frac{1}{2} [\rho_0]_b \int_{\partial S_b} g_0 u_r^2 ds \right)$$

$$- \frac{d}{dt} \left(\frac{1}{2} \int_b^a \frac{d\rho_0}{dr} \int_{\partial S_i} g_0 u_r^2 ds dr \right)$$

$$+ \frac{d}{dt} \left(\frac{1}{2} \sum_{i=a,b} [\rho_0]_i \int_{\partial S_i} u_r \Phi ds - \frac{1}{2} \int_b^a \frac{d\rho_0}{dr} \int_{\partial S_r} u_r \Phi ds dr \right)$$

= 0

Powers

$$- \int_{S_a-S_b} \boldsymbol{\tau} : \nabla \mathbf{v} dv$$

$$- \int_{S_a-S_b} \rho_0 \nabla \Psi \cdot \mathbf{v} dv - \int_{\partial S_b} (\rho_c \Psi \mathbf{v}) \cdot \mathbf{e}_r$$

$$\int_{\partial S_a} \mathbf{v} \cdot \boldsymbol{\tau} \cdot \mathbf{e}_r ds - \int_{\partial S_b} \mathbf{v} \cdot (\boldsymbol{\tau} - p_c \mathcal{I}) \cdot \mathbf{e}_r ds$$

$$- \int_{S_a-S_b} (\mathbf{u} \cdot \nabla \rho_0) \mathbf{g}_0 \cdot \mathbf{v} dv$$

$$- \int_{S_a-S_b} \rho_0 \nabla \Phi \cdot \mathbf{v} dv - \int_{\partial S_b} (\rho_c \Phi \mathbf{v}) \cdot \mathbf{e}_r$$

= 0

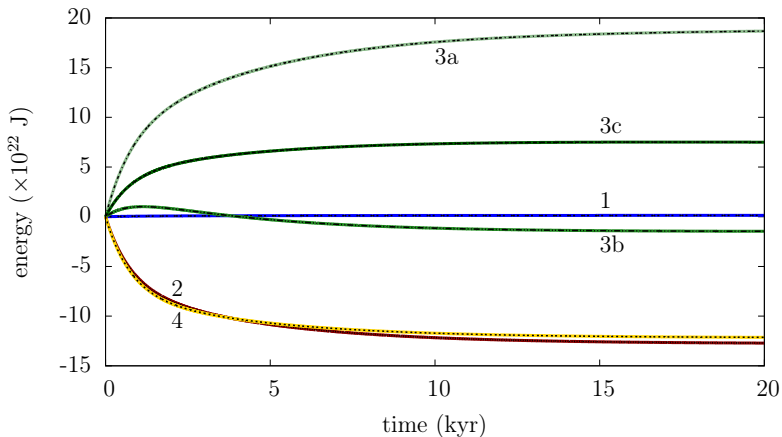
Detour: power of the centrifugal force

$$\begin{aligned}
 \int_{v(t)} \rho \nabla \Psi \cdot \mathbf{v} \, dv &= \int_{v(t)} \rho \frac{\partial \Psi}{\partial t} \, dv = \frac{1}{2} \int_{v(t)} \rho \left(\frac{\partial \Psi}{\partial t} + \nabla \Psi \cdot \mathbf{v} \right) \, dv \\
 &= \frac{1}{2} \int_{v(t)} \rho \frac{\mathcal{D} \Psi}{\mathcal{D} t} \, dv = \frac{d}{dt} \int_{v(t)} \frac{1}{2} \rho \Psi \, dv = \frac{d}{dt} \left(\frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \right)
 \end{aligned}$$

the **red** equality is based on the following:

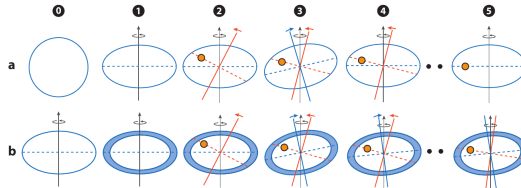
$$\begin{aligned}
 \frac{d}{dt} \int_{v(t)} \left(\rho \epsilon + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right) \, dv &= - \int_{v(t)} \rho \nabla V \cdot \mathbf{v} \, dv \\
 \frac{d}{dt} \int_{v(t)} \left(\rho \epsilon + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right) \, dv + \int_{v(t)} \rho \frac{1}{2} \frac{\mathcal{D} V}{\mathcal{D} t} \, dv &= \int_{v(t)} \rho \left(\frac{\partial V}{\partial t} - \frac{1}{2} \frac{\mathcal{D} V}{\mathcal{D} t} \right) \, dv \\
 \rho \left(\frac{\partial V}{\partial t} - \frac{1}{2} \frac{\mathcal{D} V}{\mathcal{D} t} \right) &= \rho \left(\frac{\partial V}{\partial t} - \frac{1}{2} \frac{\partial V}{\partial t} - \frac{1}{2} \nabla V \cdot \mathbf{v} \right) = \frac{1}{2} \rho \left(\frac{\partial V}{\partial t} - \mathbf{v} \cdot \nabla V \right) \\
 \frac{d}{dt} \int_{v(t)} \left(\rho \epsilon + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \rho V \right) \, dv &= \frac{1}{2} \int_{v(t)} \rho \left(\frac{\partial V}{\partial t} - \nabla V \cdot \mathbf{v} \right) \, dv
 \end{aligned}$$

Powers ↔ Energies: full Liouville equation

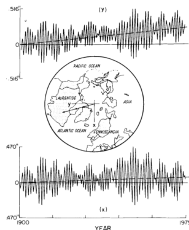


Prevailing methods

- Stationary solution



- Linearized Liouville equation



- Quasi-fluid approximation

Munk and MacDonald (1960):

$$\begin{array}{lll} I_{11} = A + j_{11} & I_{22} = A + j_{22} & I_{33} = C + j_{33} \\ I_{12} = j_{12} & J_{13} = j_{13} & I_{23} = j_{23} \\ \omega_1 = \Omega m_1 & \omega_2 = \Omega m_2 & \omega_3 = \Omega(1 + m_3) \end{array}$$

If we neglect the products and squares of $\frac{j_{ij}}{C}$, m_i , $\frac{h_i}{\Omega C}$, as they are small dimensionless numbers (for the case of the Earth), the Liouville equations decouple into two:

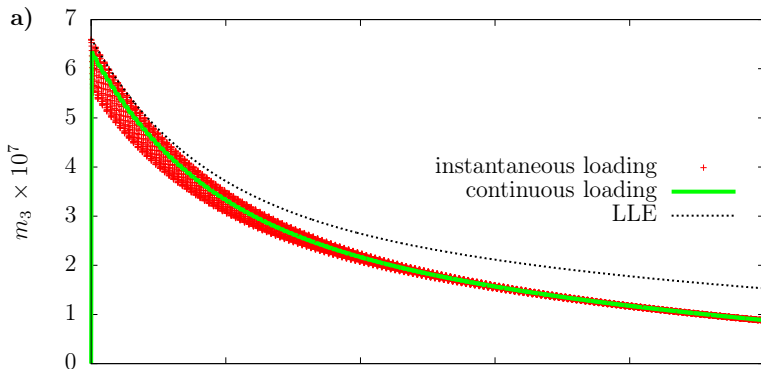
$$i \frac{\dot{\mathbf{m}}}{\sigma_r} + \mathbf{m} = \Phi - \frac{i}{\Omega} \dot{\Phi} + \frac{L}{(C - A)\Omega^2}, \quad \dot{m}_3 = \dot{\phi}_3. \quad (17)$$

The first one is called the polar motion (PM) equation, the second length of day (LOD) equation, for that is what they govern. Complex notation was introduced: $\mathbf{m} = m_1 + im_2$, $\Phi = \phi_1 + i\phi_2$.

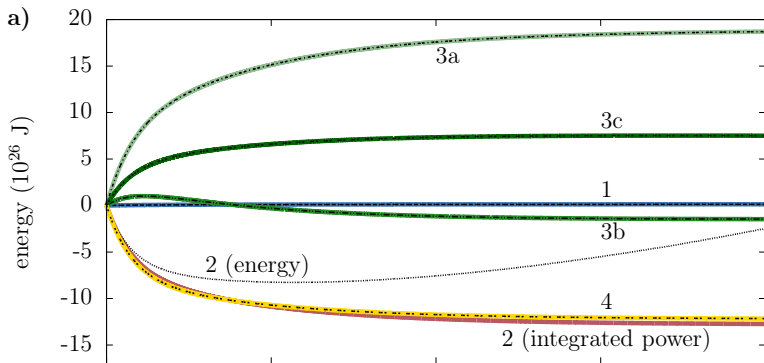
Historical note:

$$\begin{array}{ll} j_{11}^R = j_{22}^R = -\frac{2k_2^T}{3k_f}(C - A)m_3, & j_{33}^R = \frac{4k_2^T}{3k_f}(C - A)m_3 \\ j_{13}^R = \frac{k_2^T}{k_f}(C - A)m_1, & j_{23}^R = \frac{k_2^T}{k_f}(C - A)m_2 \end{array} \quad (18)$$

Linearized Liouville equation



Powers ↔ Energies: linearized Liouville equation



Outlook (patocka.vojtech@gmail.com)

✓ LLE vs full LE

- Quasi-fluid approximation vs full LE

$$J_{ij}(t) = \mathbf{I} \delta_{ij} + \frac{k_2^T(t) a^5}{3G} * [\omega_i(t) \omega_j(t) - \frac{1}{3} \omega^2(t) \delta_{ij}] + [\delta(t) + k_2^L(t)] * C_{ij}(t)$$

Symbol * denoting the time convolution. The viscoelastic Love numbers are computed in the Laplace domain:

$$k(s) = k_e + \sum_{i=1}^M \frac{k_i}{s - s_i},$$

where M is the number of relaxation modes, $\frac{-1}{s_i}$ their relaxation times and k_i their amplitudes.

$$k^T(s) = k_f^T(-T_1 s), \quad k^L(t) = k_f^L + \frac{k_{M1}^L}{s_{M1}} e^{s_{M1} t}$$

after neglecting the terms in $\ddot{\omega}$ and $\dot{\omega}^2$.. brute form of Q.F. is $\boldsymbol{\omega} \times (\mathbf{I} \cdot \boldsymbol{\omega}) = 0$

- Stationary solution vs full LE: time scales of load formation
- Planetary reorientation: transient loads